Lecture 1

Fundamentals

Course *Block-structured Adaptive Finite Volume Methods in C++*

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Outline

Outline

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Hyperbolic Conservation Laws

$$\frac{\partial}{\partial t}\mathbf{q}(\mathbf{x},t) + \sum_{n=1}^{d} \frac{\partial}{\partial x_n}\mathbf{f}_n(\mathbf{q}(\mathbf{x},t)) = 0\,, \quad D \subset \{(\mathbf{x},t) \in \mathbb{R}^d \times \mathbb{R}_0^+\} \qquad (1)$$

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$\mathbf{q} = \mathbf{q}(\mathbf{x},t) \in S \subset \mathbb{R}^M$ - vector of state, $\mathbf{f}_n(\mathbf{q}) \in \mathrm{C}^1(S,\mathbb{R}^M)$ - flux functions,

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Definition (Hyperbolicity)

$\mathbf{A}(\mathbf{q},\nu) = \nu_1\mathbf{A}_1(\mathbf{q}) + \cdots + \nu_d\mathbf{A}_d(\mathbf{q})$ with $\mathbf{A}_n(\mathbf{q}) = \partial\mathbf{f}_n(\mathbf{q})/\partial\mathbf{q}$ has M real eigenvalues $\lambda_1(\mathbf{q},\nu) \le ... \le \lambda_M(\mathbf{q},\nu)$ and M linear independent right eigenvectors $\mathbf{r}_m(\mathbf{q},\nu)$.

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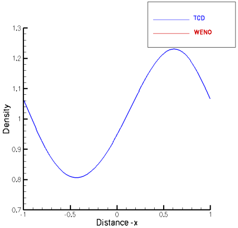
If $\mathbf{f}_n(\mathbf{q})$ is nonlinear, classical solutions $\mathbf{q}(\mathbf{x},t) \in \mathrm{C}^1(D,S)$ do not generally exist, not even for $\mathbf{q}_0(\mathbf{x}) \in \mathrm{C}^1(\mathbb{R}^d,S)$ [Majda, 1984], [Godlewski and Raviart, 1996], [Kröner, 1997]



Example: Euler equations

Characteristic variables

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and Λ the diagonal matrix of eigenvalues

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_M \end{pmatrix}$$

Characteristic variables - II

Multiplying (2) with $\mathbf{R}^{-1}$ gives

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with $\mathbf{R}^{-1}d\mathbf{q} = d\mathbf{v}$ this becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{R}^{-1}\mathbf{A}\mathbf{R}\frac{\partial \mathbf{v}}{\partial x} = 0$$

or

$$\frac{\partial \mathbf{v}}{\partial t} + \Lambda\frac{\partial \mathbf{v}}{\partial x} = 0$$

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which is just a set of decoupled independent advection equations for the components, i.e.,

$$\frac{\partial \mathbf{v}_m}{\partial t} + \lambda_m\frac{\partial \mathbf{v}_m}{\partial x} = 0 \qquad \text{for } m = 1, \ldots, M \tag{3}$$

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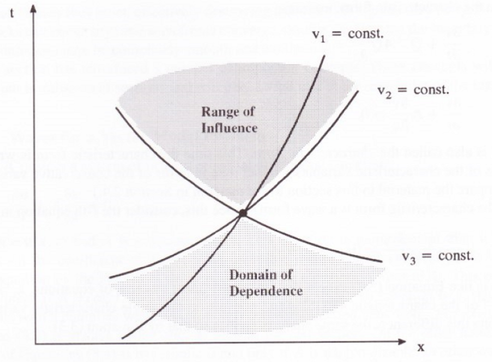
(3) is a wave equation but note that in the general quasi-linear case the eigenvalues can dependent on all v_m , i.e. $\lambda_m = \lambda_m(v_1, \cdots, v_M)$ Nevertheless, an analysis as for the wave equations shows

$$\mathbf{v}_m = const. \qquad \text{for } \frac{dx}{dt} = \lambda_m$$

Wavefronts

The curves $dx = \lambda_m \, dt$ are called wavefronts or characteristics, v_m are the characteristic variables.
The characteristics define how influence spreads in the $x - t$ plane. A point in the $x - t$ plane is only influenced by points at earlier times in a finite domain of dependence and influences only points in a finite range of influence.



Typical wave diagram for vector model problem.

Weak solutions

Integral form (Gauss's theorem):

$$\int\limits_{\Omega} \mathbf{q}(\mathbf{x}, t+\Delta t)\, d\mathbf{x} - \int\limits_{\Omega} \mathbf{q}(\mathbf{x}, t)\, d\mathbf{x} \\ + \sum_{n=1}^{d} \int\limits_{t}^{t+\Delta t} \int\limits_{\partial\Omega} \mathbf{f}_n(\mathbf{q}(\mathbf{o}, t))\, \sigma_n(\mathbf{o})\, d\mathbf{o}\, dt = \int\limits_{t}^{t+\Delta t} \int\limits_{\Omega} \mathbf{s}(\mathbf{q}(\mathbf{x}, t))\, d\mathbf{x}$$

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Theorem (Weak solution)

$\mathbf{q}_0 \in \mathrm{L}^{\infty}_{loc}(\mathbb{R}^d, S)$. $\mathbf{q} \in \mathrm{L}^{\infty}_{loc}(D, S)$ *is weak solution if* $\mathbf{q}$ *satisfies*

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$$\int_0^{\infty} \int_{\mathbb{R}^d} \left[\frac{\partial \varphi}{\partial t} \cdot \mathbf{q} + \sum_{n=1}^{d} \frac{\partial \varphi}{\partial x_n} \cdot \mathbf{f}_n(\mathbf{q}) - \varphi \cdot \mathbf{s}(\mathbf{q}) \right] d\mathbf{x}\, dt + \int_{\mathbb{R}^d} \varphi(\mathbf{x}, 0) \cdot \mathbf{q}_0(\mathbf{x})\, d\mathbf{x} = 0$$

for any test function $\varphi \in \mathrm{C}^1_0(D, S)$

Rankine-Hugoniot relations

Consider the 1d version of (1), $\mathbf{s}(\mathbf{q}) = 0$ integrated over interval $[x, x + dx] \times [t, t + dt]$

$$\int_{x}^{x+dx} \mathbf{q}(x', t + dt)dx' - \int_{x}^{x+dx} \mathbf{q}(x', t)dx' = -\int_{t}^{t+dt} \left[\mathbf{f}(\mathbf{q}(x + dx, t')) - f(\mathbf{q}(x, t'))\right] dt'$$

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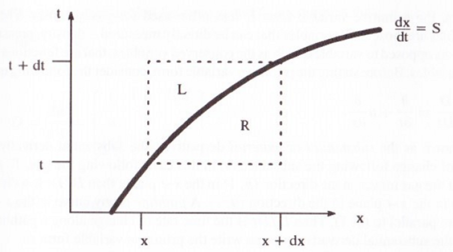
$$S = \frac{dx}{dt}$$



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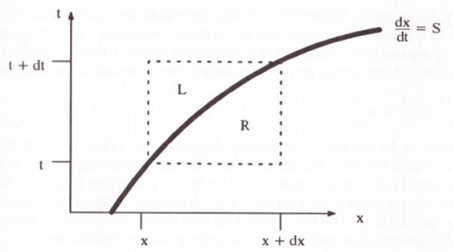
State on the left of discontinuity is index with L, on the right with R Inserting the states into (8) gives

$$(\mathbf{q}_L - \mathbf{q}_R)\, dx = -\left[\mathbf{f}(\mathbf{q}_R) - \mathbf{f}(\mathbf{q}_L)\right]\ dt$$



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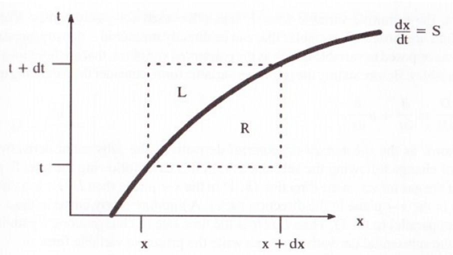

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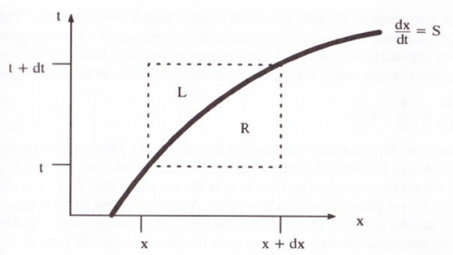

This is called Rankine-Hugoniot jump relation. Note the form $\mathbf{f}(\mathbf{q}_R) = \mathbf{f}(\mathbf{q}_L)$ for $S = 0$ from which, for instance, the shock relations for Euler equations are derived.

Entropy solutions

Select physical weak solution as $\lim_{\varepsilon \to 0} \mathbf{q}_\varepsilon = \mathbf{q}$ almost everywhere in D of

$$\frac{\partial \mathbf{q}_\varepsilon}{\partial t} + \sum_{n=1}^{d} \frac{\partial \mathbf{f}_n(\mathbf{q}_\varepsilon)}{\partial x_n} - \varepsilon \sum_{n=1}^{d} \frac{\partial^2 \mathbf{q}_\varepsilon}{\partial x_n^2} = \mathbf{s}(\mathbf{q}_\varepsilon)\,, \quad \mathbf{x} \in \mathbb{R}^d\,, \quad t > 0$$

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Assume existence of entropy $\eta \in \mathrm{C}^2(S, \mathbb{R})$ *and entropy fluxes* $\psi_n \in \mathrm{C}^1(S, \mathbb{R})$ *that satisfy*

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then $\lim_{\varepsilon\to 0} \mathbf{q}_\varepsilon = \mathbf{q}$ almost everywhere in D is weak solution and satisfies

$$\frac{\partial \eta(\mathbf{q})}{\partial t} + \sum_{n=1}^{d} \frac{\partial \psi_n(\mathbf{q})}{\partial x_n} \le \frac{\partial \eta(\mathbf{q})}{\partial \mathbf{q}}^T \cdot \mathbf{s}(\mathbf{q})$$

in the sense of distributions. Proof: [Godlewski and Raviart, 1996]

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for all entropy functions $\eta(\mathbf{q})$ and all test functions $\varphi \in \mathrm{C}_0^1(D, \mathbb{R}_0^+)$, $\varphi \geq 0$

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An entropy solution $\mathbf{q}$ *is a classical solution* $\mathbf{q} \in \mathrm{C}^1(D,S)$ *almost everywhere and satisfies the Rankine-Hugoniot (RH) jump condition*

$$(\mathbf{q}^+ - \mathbf{q}^-)\, \sigma_t + \sum_{n=1}^{d} \left(\mathbf{f}_n(\mathbf{q}^+) - \mathbf{f}_n(\mathbf{q}^-) \right) \sigma_n = \mathbf{0}$$

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and the jump inequality

$$(\eta(\mathbf{q}^+) - \eta(\mathbf{q}^-))\, \sigma_t + \sum_{n=1}^{d} \left(\psi_n(\mathbf{q}^+) - \psi_n(\mathbf{q}^-) \right) \sigma_n \leq 0$$

along discontinuities. Proof: [Godlewski and Raviart, 1996]

Euler equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_n}\left(\rho u_n\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho u_k\right) + \frac{\partial}{\partial x_n}\left(\rho u_k u_n + \delta_{kn} p\right) = 0 \,, \quad k = 1, \ldots, d$$

$$\frac{\partial}{\partial t}\left(\rho E\right) + \frac{\partial}{\partial x_n}\left(u_n(\rho E + p)\right) = 0$$

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with polytrope gas equation of state

$$p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho u_n u_n\right)$$

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$$\frac{\partial}{\partial t}\left(\rho E\right) + \frac{\partial}{\partial x_n}\left(u_n(\rho E + p)\right) = 0$$

with polytrope gas equation of state

$$p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho u_n u_n\right)$$

have structure

$$\partial_t \mathbf{q}(\mathbf{x}, t) + \nabla \cdot \mathbf{f}(\mathbf{q}(\mathbf{x}, t)) = 0$$

Characteristic form of the Euler equations

The Jacobian can be written in different forms, using

$$a^2 = \gamma \frac{p}{\rho}, \quad h = e + \frac{p}{\rho}, \quad H = h + \frac{1}{2} u^2 \quad \Rightarrow \quad H = \frac{a^2}{\gamma - 1} + \frac{1}{2} u^2$$

For

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{\gamma - 3}{2} u^2 & (3-\gamma) u & \gamma - 1 \\ -uH + \dfrac{1}{2}(\gamma - 1) u^3 & H - (\gamma - 1) u^2 & \gamma u \end{bmatrix}$$

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The matrices

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 \\ u - a & u & u + a \\ H - ua & \frac{1}{2}u^2 & H + ua \end{bmatrix}$$

$$\mathbf{R}^{-1} = \frac{1}{2a^2} \begin{bmatrix} \frac{1}{2}(\gamma - 1)u^2 + ua & (1-\gamma)\,u - a & \gamma - 1 \\ 2a^2 - (\gamma - 1)u^2 & 2(\gamma - 1)\,u & 2(1-\gamma) \\ \frac{1}{2}(\gamma - 1)u^2 - ua & (1-\gamma)\,u + a & \gamma - 1 \end{bmatrix}$$

diagonalize **A** as

Characteristic form of the Euler equations - II

$$\mathbf{R}^{-1}\,\mathbf{A}\,\mathbf{R} = \Lambda = \begin{pmatrix} u-a & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u+a \end{pmatrix}$$

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The transformation $\mathbf{R}^{-1}d\mathbf{q} = \mathbf{R}^{-1}(d\rho,\ d\left(\rho u\right),\ d\left(\rho E\right))^T$ into characteristic variables therefore leads to

$$\frac{\partial \mathbf{v}^-}{\partial t} + (u-a)\frac{\partial \mathbf{v}^-}{\partial x} = 0$$
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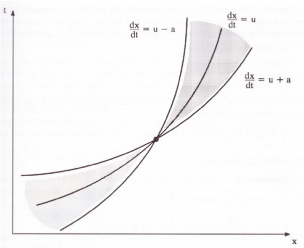
with

$$dv^- = du - \frac{dp}{\rho a} = 0 \quad \text{for} \quad dx = (u-a)dt$$
$$dv_0 = d\rho - \frac{dp}{a^2} = 0 \quad \text{for} \quad dx = u\,dt$$
$$dv^+ = du + \frac{dp}{\rho a} = 0 \quad \text{for} \quad dx = (u+a)dt$$



The crossing of characteristics causes a shock wave.

Rarefaction and shock waves in the $x - t$ plane

Consider the two enclosing characteristics $b_1(t) \leq x \leq y \leq b_2(t)$

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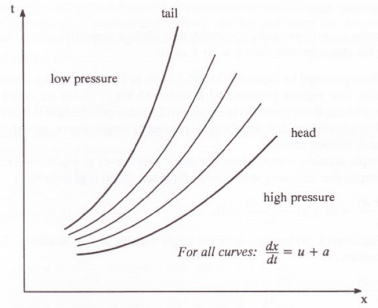

Rarefaction:

$$u(x,t) \pm a(x,t) \leq u(y,t) \pm a(y,t)$$

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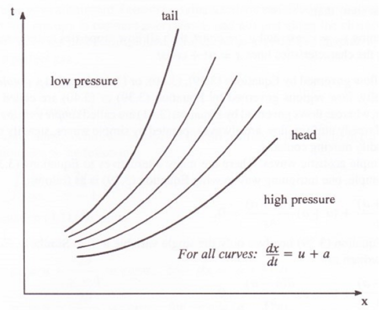

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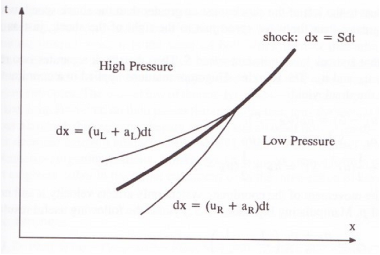

Shocks:

$$u(x,t) \pm a(x,t) \geq u(y,t) \pm a(y,t)$$

which gives for the shock speed

$$u_L \pm a_L \geq S \geq u_R \pm a_R$$

Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_n}\left(\rho u_n\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho u_k\right) + \frac{\partial}{\partial x_n}\left(\rho u_k u_n + \delta_{kn} p - \tau_{kn}\right) = 0\ , \quad k = 1, \ldots, d$$

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$$\tau_{kn} = \mu\Big(\frac{\partial u_n}{\partial x_k} + \frac{\partial u_k}{\partial x_n}\Big) - \frac{2}{3}\mu \frac{\partial u_j}{\partial x_j}\delta_{kn}$$

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Type can be either hyperbolic or parabolic

Navier-Stokes equations for multiple species

For multiple species with chemical reaction, the Navier-Stokes equations would be extended to

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x_n}(\rho_i u_n + \rho \nu_{in}) = W_i \, \dot{\omega}_i , \qquad i = 1, \ldots, N$$

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$$\nu_{in} = D_i \frac{\partial Y_i}{\partial x_n}$$

of species i into the mixture (note difference to binary diffusivities).

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$$\partial_t \mathbf{q}(\mathbf{x}, t) + \nabla \cdot \mathbf{f}(\mathbf{q}(\mathbf{x}, t)) + \nabla \cdot \mathbf{h}(\mathbf{q}(\mathbf{x}, t), \nabla \mathbf{q}(\mathbf{x}, t)) = \mathbf{s}(\mathbf{q}(\mathbf{x}, t))$$

Outline

Derivation

Assume $\partial_t \mathbf{q} + \partial_x \mathbf{f}(\mathbf{q}) + \partial_x \mathbf{h}(\mathbf{q}(\cdot, \partial_x \mathbf{q})) = \mathbf{s}(\mathbf{q})$

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Time discretization $t_n = n\Delta t$, discrete volumes
$I_j = [x_j - \frac{1}{2}\Delta x, x_j + \frac{1}{2}\Delta x[=: [x_{j-1/2}, x_{j+1/2}[$

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Using approximations $\mathbf{Q}_j(t) \approx \dfrac{1}{|I_j|} \displaystyle\int_{I_j} \mathbf{q}(\mathbf{x}, t)\, dx, \quad \mathbf{s}(\mathbf{Q}_j(t)) \approx \dfrac{1}{|I_j|} \displaystyle\int_{I_j} \mathbf{s}(\mathbf{q}(\mathbf{x}, t))\, dx$

and numerical fluxes

$$\mathbf{F}\left(\mathbf{Q}_j(t), \mathbf{Q}_{j+1}(t)\right) \approx \mathbf{f}(\mathbf{q}(x_{j+1/2}, t)), \quad \mathbf{H}\left(\mathbf{Q}_j(t), \mathbf{Q}_{j+1}(t)\right) \approx \mathbf{h}(\mathbf{q}(x_{j+1/2}, t), \nabla \mathbf{q}(x_{j+1/2}, t))$$

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yields after integration (Gauss theorem)

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For instance:

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left[\mathbf{F}\left(\mathbf{Q}_j^n, \mathbf{Q}_{j+1}^n\right) - \mathbf{F}\left(\mathbf{Q}_{j-1}^n, \mathbf{Q}_j^n\right)\right] -$$
$$\frac{\Delta t}{\Delta x}\left[\mathbf{H}\left(\mathbf{Q}_j^n, \mathbf{Q}_{j+1}^n\right) - \mathbf{H}\left(\mathbf{Q}_{j-1}^n, \mathbf{Q}_j^n\right)\right] + \Delta t \mathbf{s}(\mathbf{Q}_j^n)\, dt$$

Some classical definitions

$(2s+1)$-point difference scheme of the form

$$\mathbf{Q}_j^{n+1} = \mathcal{H}^{(\Delta t)}(\mathbf{Q}_{j-s}^n, \ldots, \mathbf{Q}_{j+s}^n)$$

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For each time τ there is a constant C_S and a value $n_0 \in \mathbb{N}$ such that $\|\mathcal{H}^{(\Delta t)}(\mathbf{Q}^n)\| \leq C_S$ for all $n\Delta t \leq \tau$, $n < n_0$

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If the local truncation error

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satisfies $\|\mathcal{L}^{(\Delta t)}(\cdot, t)\| \to 0$ as $\Delta t \to 0$

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Definition (Convergence)

If the global error $\mathcal{E}^{(\Delta t)}(\mathbf{x}, t) := \mathbf{Q}(\mathbf{x}, t) - \mathbf{q}(\mathbf{x}, t)$ satisfies $\|\mathcal{E}^{(\Delta t)}(\cdot, t)\| \to 0$ as $\Delta t \to 0$ for all admissible initial data $\mathbf{q}_0(\mathbf{x})$

Some classical definitions II

Definition (Order of accuracy)

$\mathcal{H}(\cdot)$ is accurate of order o if for all sufficiently smooth initial data $\mathbf{q}_0(\mathbf{x})$, there is a constant C_L, such that the local truncation error satisfies
$\|\mathcal{L}^{(\Delta t)}(\cdot,t)\| \leq C_L \Delta t^o$ for all $\Delta t < \Delta t_0$, $t \leq \tau$

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If $\mathcal{H}(\cdot)$ can be written in the form

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Definition (Consistency of a conservative method)

If the numerical flux satisfies $\mathbf{F}(\mathbf{q},\ldots,\mathbf{q}) = \mathbf{f}(\mathbf{q})$ for all $\mathbf{q} \in S$

Splitting methods

Solve homogeneous PDE and ODE successively!

$$\mathcal{H}^{(\Delta t)}: \quad \partial_t \mathbf{q} + \nabla \cdot \mathbf{f}(\mathbf{q}) = 0\,, \quad \text{IC: } \mathbf{Q}(t_m) \stackrel{\Delta t}{\Longrightarrow} \tilde{\mathbf{Q}}$$

$$\mathcal{S}^{(\Delta t)}: \quad \partial_t \mathbf{q} = \mathbf{s}(\mathbf{q})\,, \quad \text{IC: } \tilde{\mathbf{Q}} \stackrel{\Delta t}{\Longrightarrow} \mathbf{Q}(t_m + \Delta t)$$

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$$\begin{aligned} \mathcal{H}^{(\Delta t)}: &\quad \partial_t \mathbf{q} + \nabla \cdot \mathbf{f}(\mathbf{q}) = 0\,, &\quad \text{IC: } \mathbf{Q}(t_m) \stackrel{\Delta t}{\Longrightarrow} \tilde{\mathbf{Q}} \\ \mathcal{S}^{(\Delta t)}: &\quad \partial_t \mathbf{q} = \mathbf{s}(\mathbf{q})\,, &\quad \text{IC: } \tilde{\mathbf{Q}} \stackrel{\Delta t}{\Longrightarrow} \mathbf{Q}(t_m + \Delta t) \end{aligned}$$

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1st-order dimensional splitting for $\mathcal{H}^{(\cdot)}$:

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\mathcal{X}_2^{(\Delta t)}: &\quad \partial_t \mathbf{q} + \partial_{x_2} \mathbf{f}_2(\mathbf{q}) = 0\,, &\quad \text{IC: } \tilde{\mathbf{Q}}^{1/2} &\stackrel{\Delta t}{\Longrightarrow} \tilde{\mathbf{Q}}
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[Toro, 1999]

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Von Neumann stability analysis: Insert single eigenmode $\hat{Q}(t)e^{ik_1x_1}e^{ik_2x_2}$ into discretization

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i.e.

$$|1 - 4C_1 - 4C_2| \le 1$$

from which we derive the stability condition

$$0 \le c\left(\frac{\Delta t}{\Delta x_1^2} + \frac{\Delta t}{\Delta x_2^2}\right) \le \frac{1}{2}$$

Outline

The Riemann problem in the linear case

Consider the linear hyperbolic equation (i.e. $\mathbf{A} = \mathit{const.}$)

$$\frac{\partial}{\partial t}\mathbf{q}(x,t) + \mathbf{A}\frac{\partial}{\partial x}\mathbf{q}(x,t) = 0$$

Assume (for simplicity) that $\mathbf{A}$ has M distinct real eigenvalues $\lambda_1 < \ldots < \lambda_M$ with M linear independent right eigenvectors $\mathbf{r}_m$.

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We can readily apply the characteristic transformation $\mathbf{R}^{-1}\mathbf{q} = \mathbf{v}$ to obtain

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{\Lambda}\frac{\partial \mathbf{v}}{\partial x} = 0$$

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$$\frac{\partial v_m}{\partial t} + \lambda_m \frac{\partial v_m}{\partial x} = 0 \qquad \text{for all} \quad m = 1, \ldots, M$$

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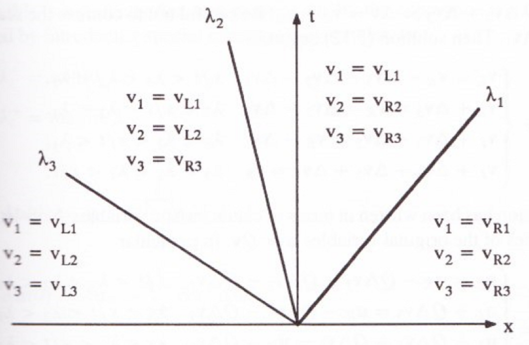
Each characteristic variable v_m changes only across the characteristic line associated to λ_m.
Since the entire problem is linear, we can simply sum up all these jumps Δv_m successively to connect the RP states $\mathbf{v}_L, \mathbf{v}_R$ in characteristic variables.

The Riemann problem in the linear case - II

Example of a linear 3 PDE system:



Introducing the jumps

$$\Delta \mathbf{v}_1 = [v_{R1} - v_{L1}, 0, 0]^T$$

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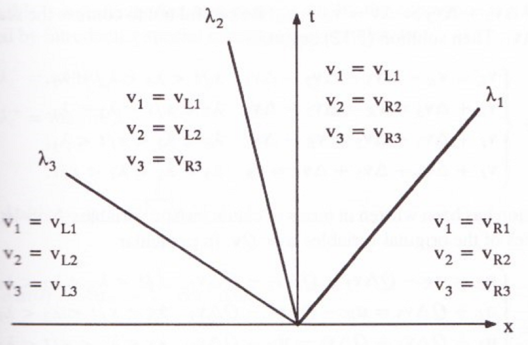

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$$\mathbf{v}\left(\frac{x}{t}\right) = \begin{cases} \mathbf{v}_L = \mathbf{v}_R - \Delta\mathbf{v}_3 - \Delta\mathbf{v}_2 - \Delta\mathbf{v}_1 & x/t < \lambda_3 \\ \mathbf{v}_L + \Delta\mathbf{v}_3 = \mathbf{v}_R - \Delta\mathbf{v}_2 - \Delta\mathbf{v}_1 & \lambda_3 < x/t < \lambda_2 \\ \mathbf{v}_L + \Delta\mathbf{v}_3 + \Delta\mathbf{v}_2 = \mathbf{v}_R - \Delta\mathbf{v}_1 & \lambda_2 < x/t < \lambda_1 \\ \mathbf{v}_L + \Delta\mathbf{v}_3 + \Delta\mathbf{v}_2 + \Delta\mathbf{v}_1 = \mathbf{v}_R & \lambda_1 < x/t \end{cases}$$

The Riemann problem in the linear case - III

or using the transformation $\mathbf{q} = \mathbf{R}\mathbf{v}$ and $\mathbf{R}\Delta\mathbf{v}_m = r_m \Delta v_m$

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Multiplying with $\mathbf{A}$ and using $\mathbf{A}\mathbf{r}_m = \lambda_m \mathbf{r}_m$ gives

$$\mathbf{A}\mathbf{q}\left(\frac{x}{t}\right) = \begin{cases} \mathbf{A}\mathbf{q}_L = \mathbf{A}\mathbf{q}_R - \mathbf{r}_3\lambda_3\Delta v_3 - \mathbf{r}_2\lambda_2\Delta v_2 - \mathbf{r}_1\lambda_1\Delta v_1 & x/t < \lambda_3 \\ \mathbf{A}\mathbf{q}_L + \mathbf{r}_3\lambda_3\Delta v_3 = \mathbf{A}\mathbf{q}_R - \mathbf{r}_2\lambda_2\Delta v_2 - \mathbf{r}_1\lambda_1\Delta v_1 & \lambda_3 < x/t < \lambda_2 \\ \mathbf{A}\mathbf{q}_L + \mathbf{r}_3\lambda_3\Delta v_3 + \mathbf{r}_2\lambda_2\Delta v_2 = \mathbf{A}\mathbf{q}_R - \mathbf{r}_1\lambda_1\Delta v_1 & \lambda_2 < x/t < \lambda_1 \\ \mathbf{A}\mathbf{q}_L + \mathbf{r}_3\lambda_3\Delta v_3 + \mathbf{r}_2\lambda_2\Delta v_2 + \mathbf{r}_1\lambda_1\Delta v_1 = \mathbf{A}\mathbf{q}_R & \lambda_1 < x/t \end{cases}$$

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This allows direct evaluation of the flux at x=0.

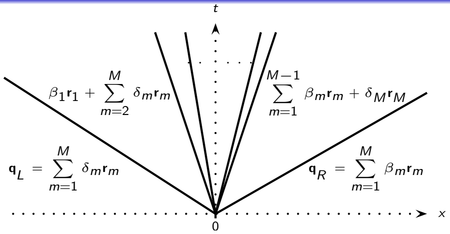
$$\mathbf{F}(\mathbf{q}_{_L}, \mathbf{q}_{_R}) := \mathbf{f}(\mathbf{q}(0,t)) = \mathbf{A}\,\mathbf{q}(0,t)$$

Linear upwind schemes

Consider Riemann problem

$$\frac{\partial}{\partial t}\mathbf{q}(x,t)+\mathbf{A}\frac{\partial}{\partial x}\mathbf{q}(x,t)=\mathbf{0}\,,\quad x\in\mathbb{R}\,,\quad t>0$$

Linear upwind schemes



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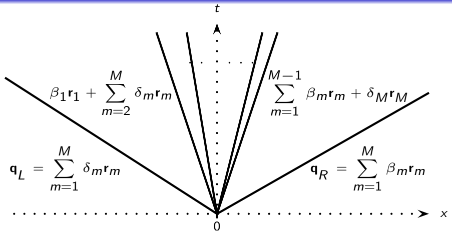

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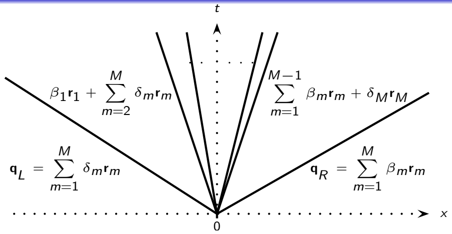

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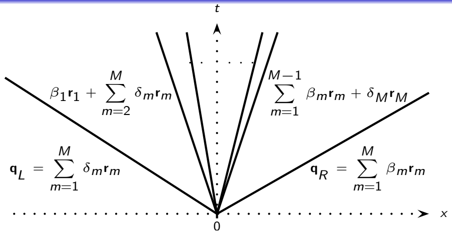
Use $\qquad \lambda_m^+=\max(\lambda_m,0)\,,\qquad \lambda_m^-=\min(\lambda_m,0)$

to define $\qquad \mathbf{\Lambda}^+:=\mathrm{diag}(\lambda_1^+,\dots,\lambda_M^+)\,,\qquad \mathbf{\Lambda}^-:=\mathrm{diag}(\lambda_1^-,\dots,\lambda_M^-)$

Linear upwind schemes



Consider Riemann problem

$$\frac{\partial}{\partial t}\mathbf{q}(x,t) + \mathbf{A}\frac{\partial}{\partial x}\mathbf{q}(x,t) = \mathbf{0}\,, \quad x \in \mathbb{R}, \quad t > 0$$

Has exact solution

$$\mathbf{q}(x,t) = \mathbf{q}_L + \sum_{\lambda_m < x/t} a_m \mathbf{r}_m = \mathbf{q}_R - \sum_{\lambda_m \geq x/t} a_m \mathbf{r}_m = \sum_{\lambda_m \geq x/t} \delta_m \mathbf{r}_m + \sum_{\lambda_m < x/t} \beta_m \mathbf{r}_m$$

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$$\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{A}\mathbf{q}_L + \sum_{\lambda_m < 0} a_m \lambda_m \mathbf{r}_m = \mathbf{A}\mathbf{q}_R - \sum_{\lambda_m \geq 0} a_m \lambda_m \mathbf{r}_m = \sum_{\lambda_m \geq 0} \delta_m \lambda_m \mathbf{r}_m + \sum_{\lambda_m < 0} \beta_m \lambda_m \mathbf{r}_m$$

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and $\quad \mathbf{A}^+ := \mathbf{R}\,\mathbf{\Lambda}^+\,\mathbf{R}^{-1}\,, \qquad \mathbf{A}^- := \mathbf{R}\,\mathbf{\Lambda}^-\,\mathbf{R}^{-1} \qquad$ which gives

$$\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{A}\mathbf{q}_L + \mathbf{A}^-\Delta\mathbf{q} = \mathbf{A}\mathbf{q}_R - \mathbf{A}^+\Delta\mathbf{q} = \mathbf{A}^+\mathbf{q}_L + \mathbf{A}^-\mathbf{q}_R$$

with $\Delta\mathbf{q} = \mathbf{q}_R - \mathbf{q}_L$

Flux difference splitting

Godunov-type scheme with $\Delta \mathbf{Q}^n_{j+1/2} = \mathbf{Q}^n_{j+1} - \mathbf{Q}^n_j$

$$\mathbf{Q}^{n+1}_j = \mathbf{Q}^n_j - \frac{\Delta t}{\Delta x}\left(\mathbf{A}^- \Delta \mathbf{Q}^n_{j+1/2} + \mathbf{A}^+ \Delta \mathbf{Q}^n_{j-1/2}\right)$$

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Use linearization $\bar{\mathbf{f}}(\bar{\mathbf{q}}) = \hat{\mathbf{A}}(\mathbf{q}_L, \mathbf{q}_R)\bar{\mathbf{q}}$ and construct scheme for nonlinear problem as

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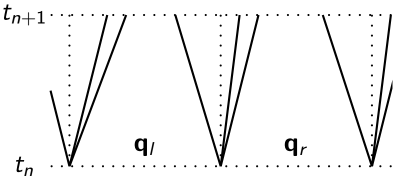
stability condition

$$\max_{j\in\mathbb{Z}} |\hat{\lambda}_{m,j+\frac{1}{2}}| \frac{\Delta t}{\Delta x} \le 1\ , \quad \text{for all } m = 1, \ldots, M$$

[LeVeque, 1992]

Roe's approximate Riemann solver

Choosing $\hat{\mathbf{A}}(\mathbf{q}_L, \mathbf{q}_R)$ [Roe, 1981]:



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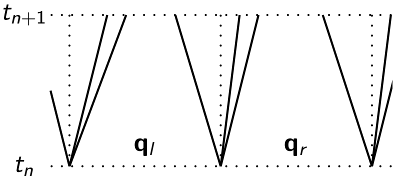

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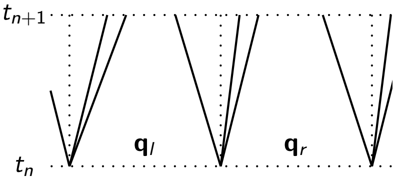

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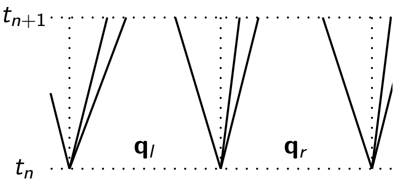

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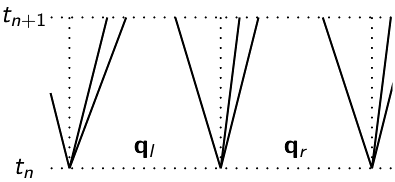

Wave decomposition: $\Delta\mathbf{q} = \mathbf{q}_r - \mathbf{q}_l = \sum_m a_m \hat{\mathbf{r}}_m$

$$\begin{aligned}\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) &= \mathbf{f}(\mathbf{q}_L) + \sum_{\hat{\lambda}_m<0} \hat{\lambda}_m \, a_m \, \hat{\mathbf{r}}_m = \mathbf{f}(\mathbf{q}_R) - \sum_{\hat{\lambda}_m\geq 0} \hat{\lambda}_m \, a_m \, \hat{\mathbf{r}}_m \\ &= \frac{1}{2}\left(\mathbf{f}(\mathbf{q}_L) + \mathbf{f}(\mathbf{q}_R) - \sum_m |\hat{\lambda}_m| \, a_m \, \hat{\mathbf{r}}_m\right)\end{aligned}$$

The Roe solver for Euler equations

For Euler equations, the following non-apparent average defines the Roe method:
The average for u and H read

$$\hat{u} := \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \qquad \hat{H} := \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

The average of the density is

$$\hat{\rho} = \frac{\sqrt{\rho_L}\rho_R + \sqrt{\rho_R}\rho_L}{\sqrt{\rho_L} + \sqrt{\rho_R}} = \sqrt{\rho_L \rho_R}$$

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and the averaged speed of sound is

$$\hat{a} := \left((\gamma - 1)(\hat{H} - \frac{1}{2}\hat{u}^2) \right)^{1/2}$$

The eigenvectors read

$$\hat{\mathbf{r}}_1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{a} \\ \hat{H} - \hat{u}\hat{a} \end{bmatrix}, \quad \hat{\mathbf{r}}_2 = \begin{bmatrix} 1 \\ \hat{u} \\ \hat{u}^2/2 \end{bmatrix}, \quad \hat{\mathbf{r}}_3 = \begin{bmatrix} 1 \\ \hat{u} + \hat{a} \\ \hat{H} + \hat{u}\hat{a} \end{bmatrix}$$

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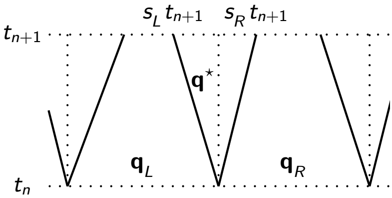
and the characteristic wave strengths are

$$\Delta v_1 = a_1 = \frac{\Delta p - \hat{\rho}\hat{a}\Delta u}{2\hat{a}^2}, \qquad \Delta v_2 = a_2 = \Delta\rho - \frac{\Delta p}{\hat{a}^2}, \qquad \Delta v_3 = a_3 = \frac{\Delta p + \hat{\rho}\hat{a}\Delta u}{2\hat{a}^2}.$$

Harten-Lax-Van Leer (HLL) approximate Riemann solver



$$\bar{\mathbf{q}}(x,t) = \begin{cases} \mathbf{q}_L\,, & x < s_L\, t \\ \mathbf{q}^\star\,, & s_L\, t \le x \le s_R\, t \\ \mathbf{q}_R\,, & x > s_R\, t \end{cases}$$

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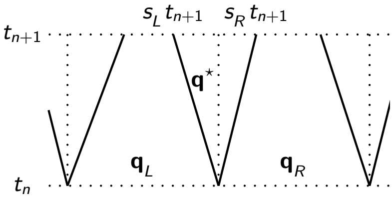

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$$\mathbf{F}_{HLL}(\mathbf{q}_L,\mathbf{q}_R) = \begin{cases} \mathbf{f}(\mathbf{q}_L)\,, & 0 < s_L\,, \\ \dfrac{s_R\mathbf{f}(\mathbf{q}_L) - s_L\mathbf{f}(\mathbf{q}_R) + s_L s_R(\mathbf{q}_R - \mathbf{q}_L)}{s_R - s_L}\,, & s_L \le 0 \le s_R\,, \\ \mathbf{f}(\mathbf{q}_R)\,, & 0 > s_R\,, \end{cases}$$

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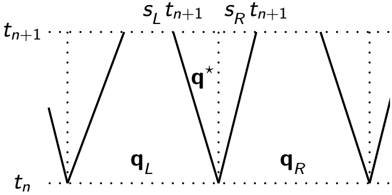

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Euler equations:

$$s_L = \min(u_{1,L} - c_L, u_{1,R} - c_R)\,, \quad s_R = \max(u_{1,L} + c_l, u_{1,R} + c_R)$$

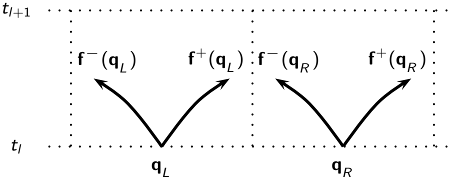
[Toro, 1999], HLLC: [Toro et al., 1994]

Flux vector splitting

Splitting

$$\mathbf{f}(\mathbf{q}) = \mathbf{f}^+(\mathbf{q}) + \mathbf{f}^-(\mathbf{q})$$

$$\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}^+(\mathbf{q}_L) + \mathbf{f}^-(\mathbf{q}_R)$$



Flux vector splitting

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$$\mathbf{f}(\mathbf{q}) = \mathbf{f}^+(\mathbf{q}) + \mathbf{f}^-(\mathbf{q})$$

derived under restriction $\hat{\lambda}_m^+ \geq 0$ and $\hat{\lambda}_m^- \leq 0$ for all $m = 1, \ldots, M$ for

$$\hat{\mathbf{A}}^+(\mathbf{q}) = \frac{\partial \mathbf{f}^+(\mathbf{q})}{\partial \mathbf{q}}, \quad \hat{\mathbf{A}}^-(\mathbf{q}) = \frac{\partial \mathbf{f}^-(\mathbf{q})}{\partial \mathbf{q}}$$

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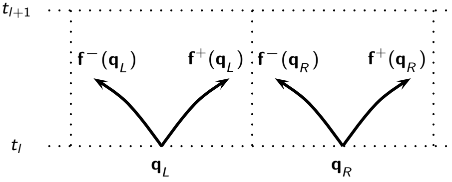

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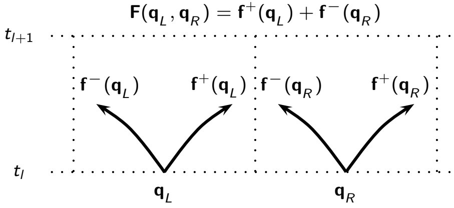

plus reproduction of regular upwinding

$$\begin{array}{lllllllll}
\mathbf{f}^{+}(\mathbf{q}) & = & \mathbf{f}(\mathbf{q}), & \mathbf{f}^{-}(\mathbf{q}) & = & \mathbf{0} & \text{if} & \lambda_m \geq 0 & \text{for all} \quad m = 1, \ldots, M \\
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\end{array}$$

Then use

$$\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}^{+}(\mathbf{q}_L) + \mathbf{f}^{-}(\mathbf{q}_R)$$

Steger-Warming

Required $\mathbf{f}(\mathbf{q}) = \mathbf{A}(\mathbf{q})\,\mathbf{q}$

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$$\mathbf{F}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{A}^+(\mathbf{q}_L)\,\mathbf{q}_L + \mathbf{A}^-(\mathbf{q}_R)\,\mathbf{q}_R$$

Jacobians of the split fluxes are identical to $\mathbf{A}^\pm(\mathbf{q})$ only in linear case

$$\frac{\partial \mathbf{f}^\pm(\mathbf{q})}{\partial \mathbf{q}} = \frac{\partial\left(\mathbf{A}^\pm(\mathbf{q})\,\mathbf{q}\right)}{\partial \mathbf{q}} = \mathbf{A}^\pm(\mathbf{q}) + \frac{\partial \mathbf{A}^\pm(\mathbf{q})}{\partial \mathbf{q}}\,\mathbf{q}$$

Further methods: Van Leer FVS [Toro, 1999], AUSM [Wada and Liou, 1997]

Steger-Warming FVS for Euler equations

For Euler equations, $\mathbf{f}(\mathbf{q}) = \mathbf{A}(\mathbf{q})\,\mathbf{q}$ holds true. We also know all matrices $\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \mathbf{\Lambda}$.

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Approach 2: Analyze sign of eigenvalues:
$u < -a$: All eigenvalues are negative: $u - a < u < u + a < 0$

$$\mathbf{f}^-(\mathbf{q}) = \left[\rho u, \rho u^2 + p, \rho u H\right]^T, \qquad \mathbf{f}^+(\mathbf{q}) = 0$$

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$u > a$: All eigenvalues are positive: $0 < u - a < u < u + a$

$$\mathbf{f}^-(\mathbf{q}) = 0, \qquad \mathbf{f}^+(\mathbf{q}) = \left[\rho u, \rho u^2 + p, \rho u H\right]^T$$

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Approach 2: Analyze sign of eigenvalues:
$u < -a$: All eigenvalues are negative: $u - a < u < u + a < 0$

$$\mathbf{f}^-(\mathbf{q}) = \left[\rho u, \rho u^2 + p, \rho u H\right]^T, \qquad \mathbf{f}^+(\mathbf{q}) = 0$$

$u > a$: All eigenvalues are positive: $0 < u - a < u < u + a$

$$\mathbf{f}^-(\mathbf{q}) = 0, \qquad \mathbf{f}^+(\mathbf{q}) = \left[\rho u, \rho u^2 + p, \rho u H\right]^T$$

$-a \le u \le a$: We find $u - a \le 0$ and $u + a \ge 0$ are always satisfied. For $u < 0$, we need to evaluate

$$\mathbf{f}^-(\mathbf{q}) = \mathbf{R}\begin{bmatrix} u-a & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{R}^{-1}\mathbf{q}, \qquad \mathbf{f}^+(\mathbf{q}) = \mathbf{R}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u+a \end{bmatrix}\mathbf{R}^{-1}\mathbf{q}$$

For $u \ge 0$, we need to evaluate

$$\mathbf{f}^-(\mathbf{q}) = \mathbf{R}\begin{bmatrix} u-a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{R}^{-1}\mathbf{q}, \qquad \mathbf{f}^+(\mathbf{q}) = \mathbf{R}\begin{bmatrix} 0 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u+a \end{bmatrix}\mathbf{R}^{-1}\mathbf{q}$$

References I

[Godlewski and Raviart, 1996] Godlewski, E. and Raviart, P.-A. (1996). *Numerical approximation of hyperbolic systems of conservation laws*. Springer Verlag, New York.

[Kröner, 1997] Kröner, D. (1997). *Numerical schemes for conservation laws*. John Wiley & Sons and B. G. Teubner, New York, Leipzig.

[LeVeque, 1992] LeVeque, R. J. (1992). *Numerical methods for conservation laws*. Birkhäuser, Basel.

[Majda, 1984] Majda, A. (1984). *Compressible fluid flow and systems of conservation laws in several space variables*. Applied Mathematical Sciences Vol. 53. Springer-Verlag, New York.

[Roe, 1981] Roe, P. L. (1981). Approximate Riemann solvers, parameter vectors and difference schemes. *J. Comput. Phys.*, 43:357–372.

[Toro, 1999] Toro, E. F. (1999). *Riemann solvers and numerical methods for fluid dynamics*. Springer-Verlag, Berlin, Heidelberg, 2nd edition.

[Toro et al., 1994] Toro, E. F., Spruce, M., and Speares, W. (1994). Restoration of the contact surface in the HLL-Riemann solver. *Shock Waves*, 4:25–34.

References II

[Wada and Liou, 1997] Wada, Y. and Liou, M.-S. (1997). An accurate and robust flux splitting scheme for shock and contact discontinuities. *SIAM J. Sci. Comp.*, 18(3):633–657.