Lecture 2

Structured adaptive mesh refinement

Course *Block-structured Adaptive Finite Volume Methods in C++*

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Outline

Meshes and adaptation

- Adaptivity on unstructured and structured meshes
- Available SAMR software

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The serial Berger-Colella SAMR method
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- Domain decomposition
- A parallel SAMR algorithm

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Elements of adaptive algorithms

- Base grid

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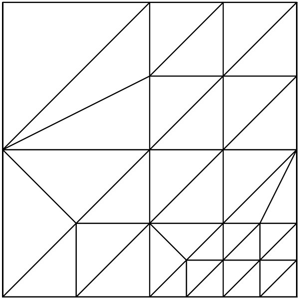
Adaptivity on unstructured meshes

- Coarse cells replaced by finer ones



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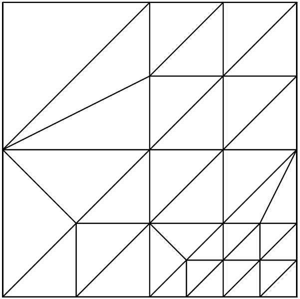

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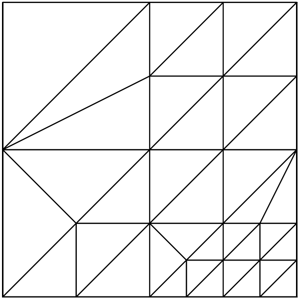

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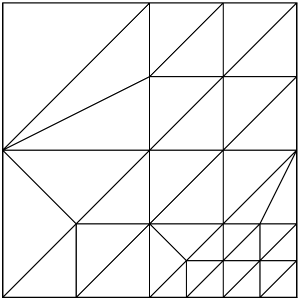

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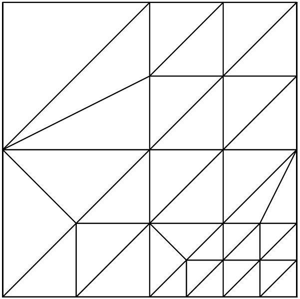

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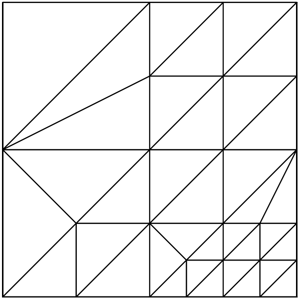

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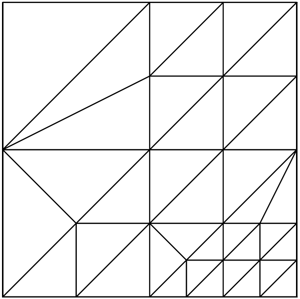

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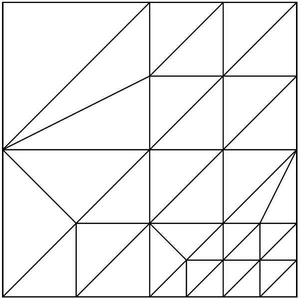

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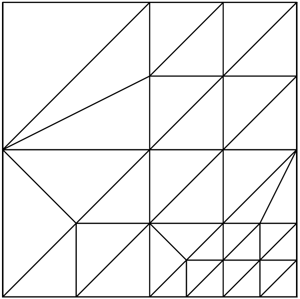

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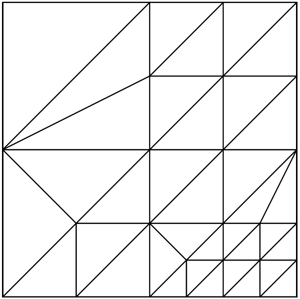

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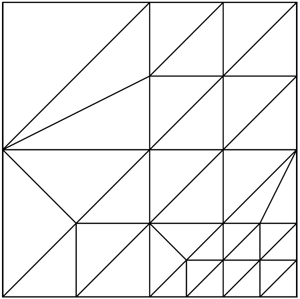

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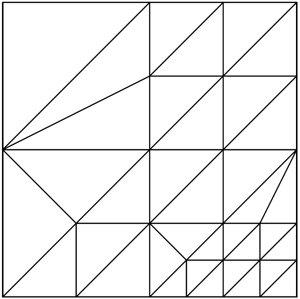

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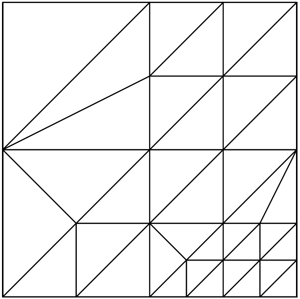

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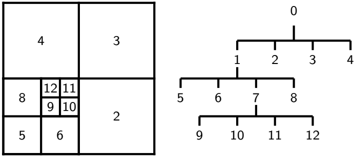

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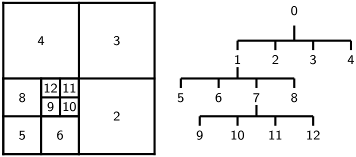

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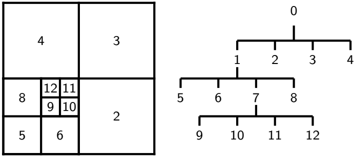

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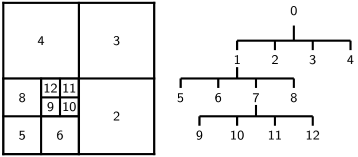

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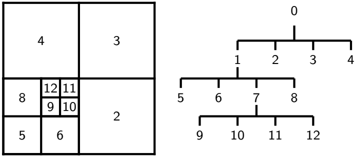

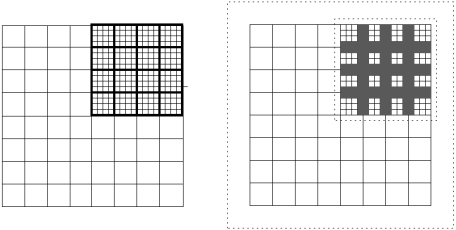

Wasted boundary space in a quad-tree

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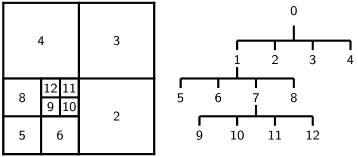

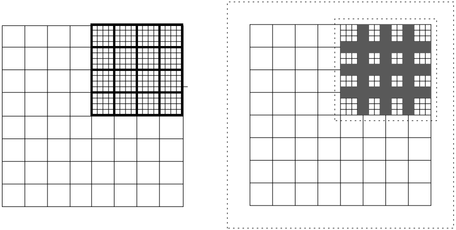

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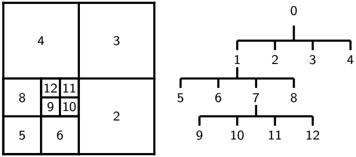

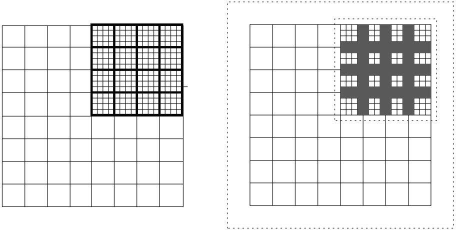

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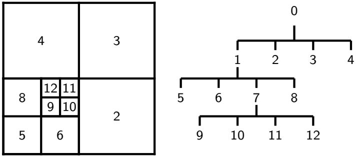

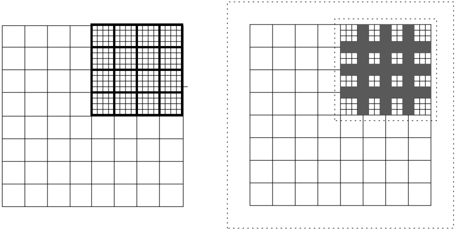

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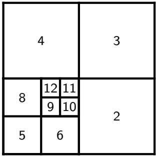

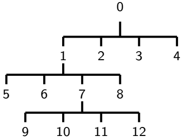

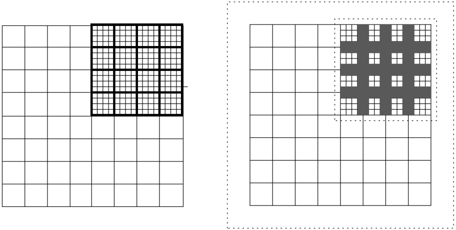

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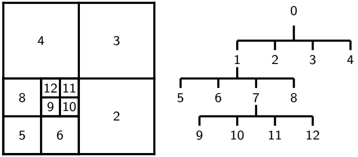
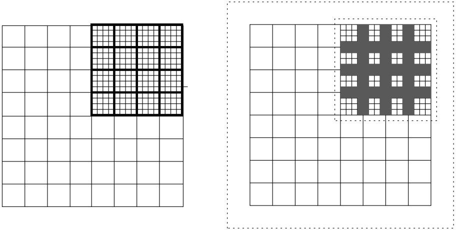
- \+ Parent/Child relations according to tree
- +/- Cache-reuse / vectorization only in data block





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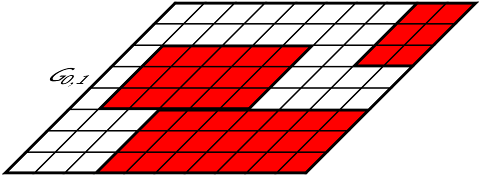
- Refined block overlay coarser ones



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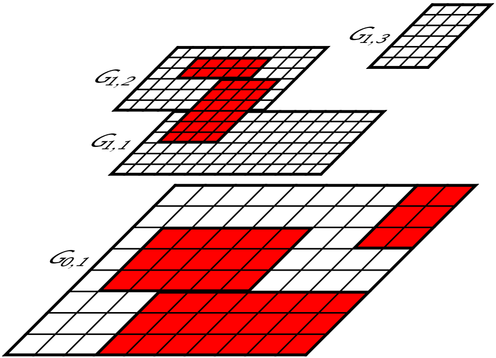

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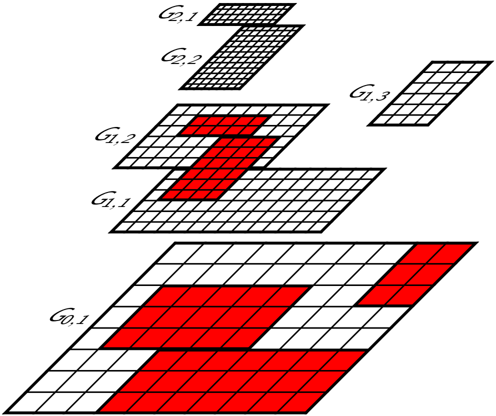

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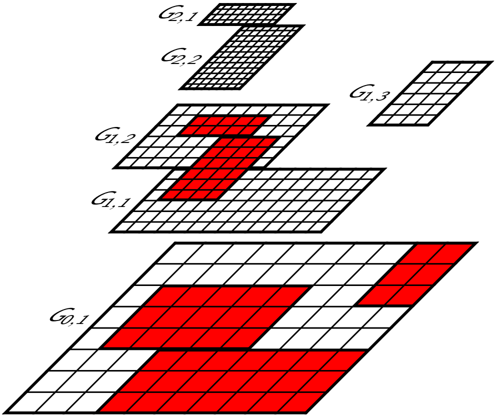

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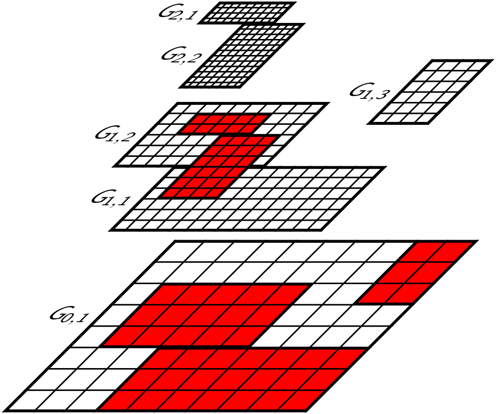

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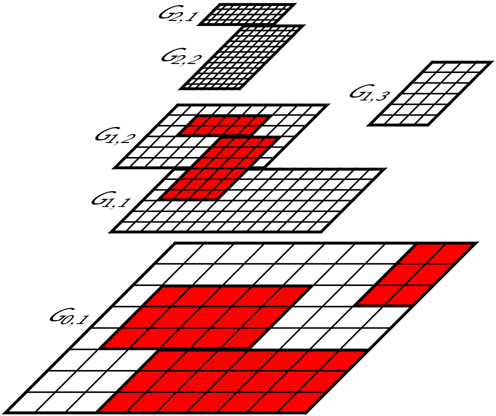

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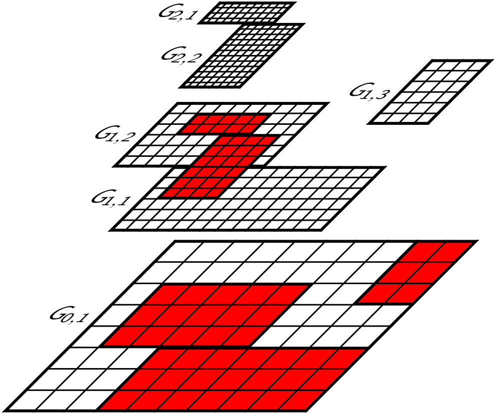

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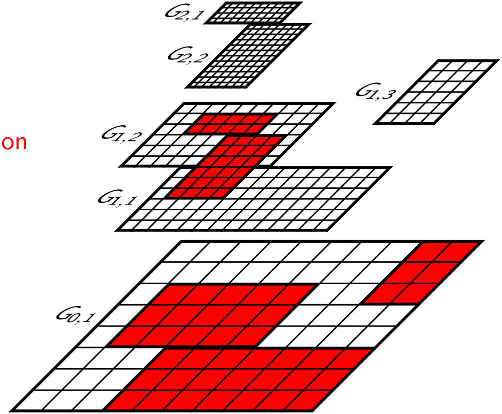

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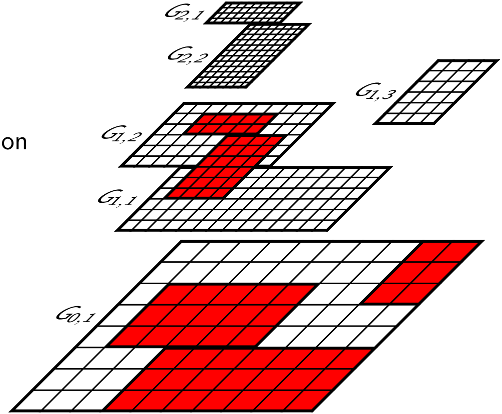

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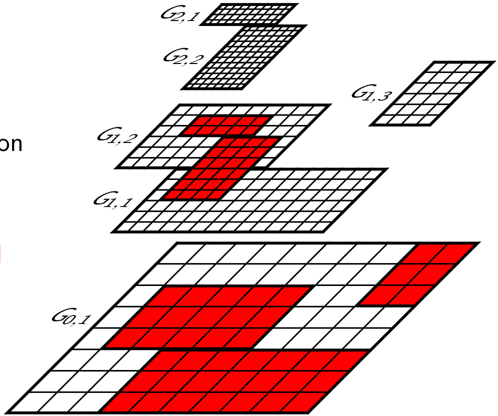

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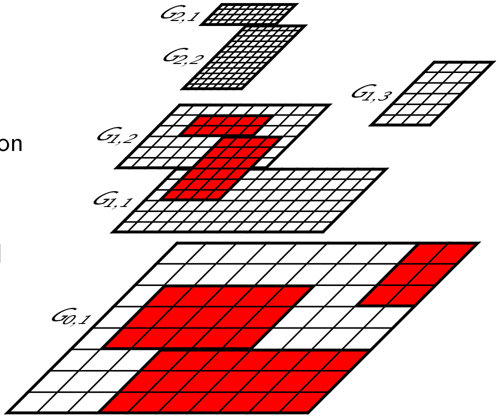

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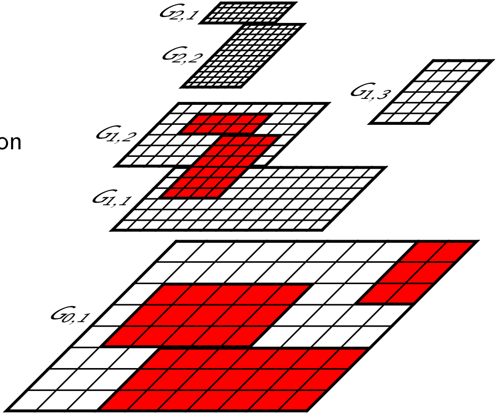

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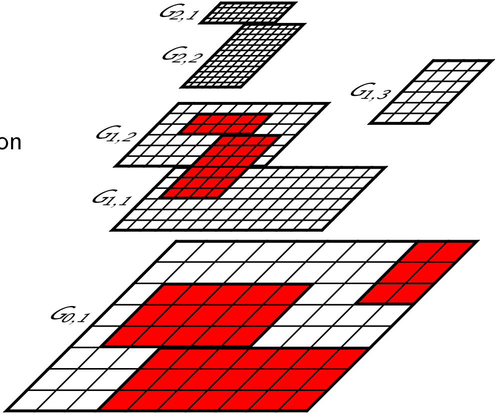

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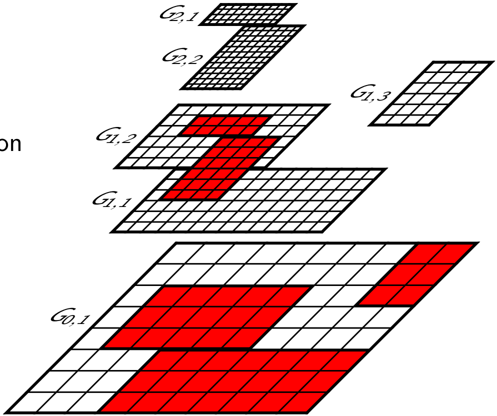
\- Difficult to implement



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- Uintah (AMR code for simulation of accidental fires and explosions)
  - Only explicit algorithms considered
  - FSI coupling Material Point Method and ICE Method (Implicit, Continuous fluid, Eulerian)
  - http://www.uintah.utah.edu

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Distributed memory parallelization fully supported if not otherwise states.

- PARAMESH (Parallel Adaptive Mesh Refinement)
    - Library based on uniform refinement blocks [MacNeice et al., 2000]
    - Both multigrid and explicit algorithms considered
    - http://sourceforge.net/projects/paramesh
- Flash code (AMR code for astrophysical thermonuclear flashes)
    - Built on PARAMESH
    - Solves the magneto-hydrodynamic equations with self-gravitation
    - http://www.flash.uchicago.edu/site/flashcode
- Uintah (AMR code for simulation of accidental fires and explosions)
    - Only explicit algorithms considered
    - FSI coupling Material Point Method and ICE Method (Implicit, Continuous fluid, Eulerian)
    - http://www.uintah.utah.edu
- DAGH/Grace [Parashar and Browne, 1997]
    - Just C++ data structures but no methods
    - All grids are aligned to bases mesh coarsened by factor 2
    - http://userweb.cs.utexas.edu/users/dagh

Systems that support general SAMR

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- SAMRAI - Structured Adaptive Mesh Refinement Application Infrastructure
  - Very mature SAMR system [Hornung et al., 2006]
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- Chombo
  - Redesign and extension of BoxLib by P. Colella et al.
  - Both multigrid and explicit algorithms demonstrated
  - Some embedded boundary support
  - https://commons.lbl.gov/display/chombo

Further SAMR software

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- Overture (Object-oriented tools for solving PDEs in complex geometries)
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- Cell-based Cartesian AMR: RAGE
  - Embedded boundary method
  - Explicit and implicit algorithms
  - [Gittings et al., 2008]

Outline

Meshes and adaptation
- Adaptivity on unstructured and structured meshes
- Available SAMR software

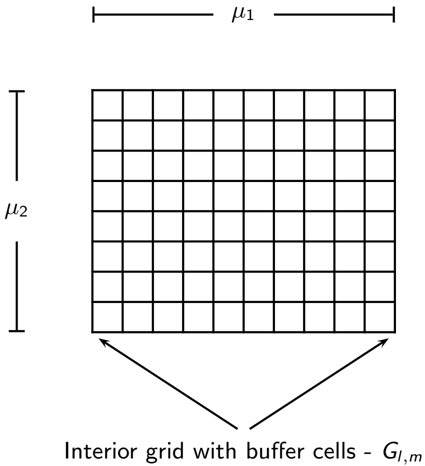
The serial Berger-Colella SAMR method
- Data structures and numerical update
- Conservative flux correction
- Level transfer operators
- The basic recursive algorithm
- Block generation and flagging of cells

Parallel SAMR method
- Domain decomposition
- A parallel SAMR algorithm

The mth refinement grid on level l

Notations:

- Boundary: $\partial G_{l,m}$



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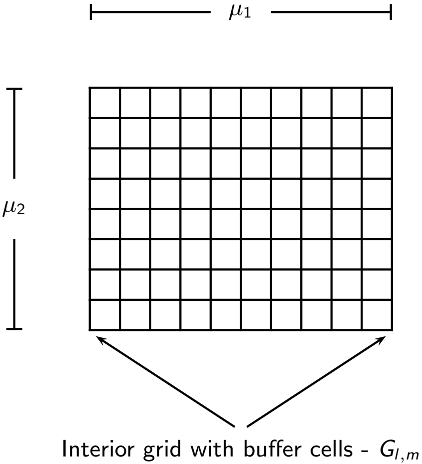

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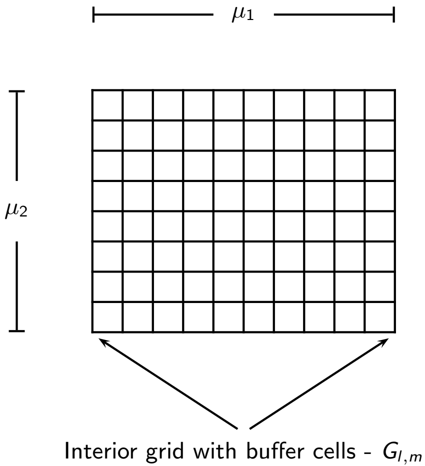

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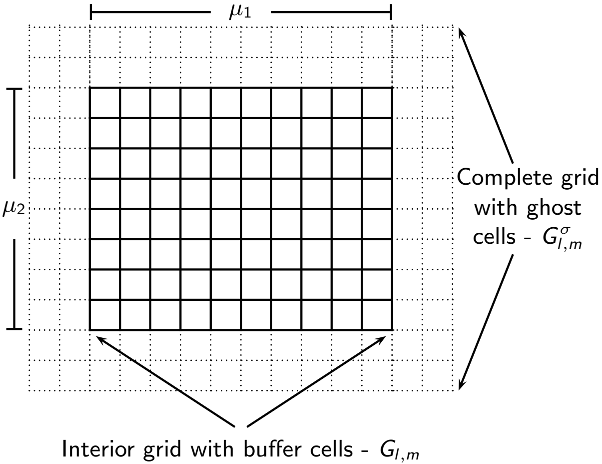

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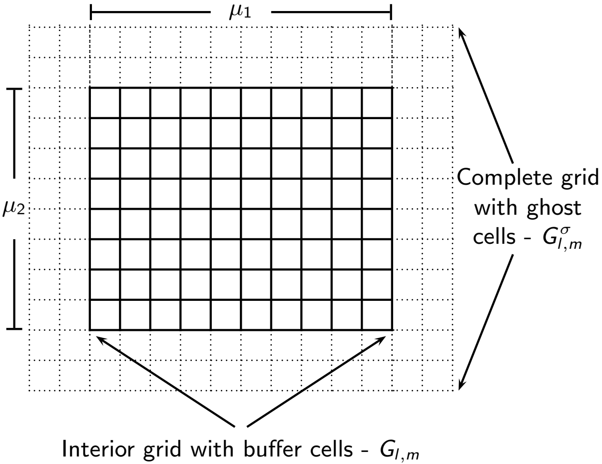

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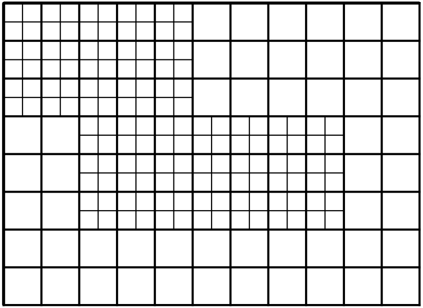
Setting of ghost cells



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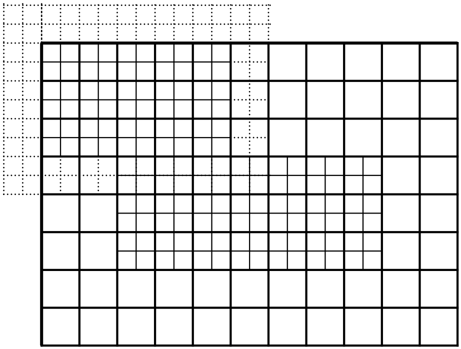

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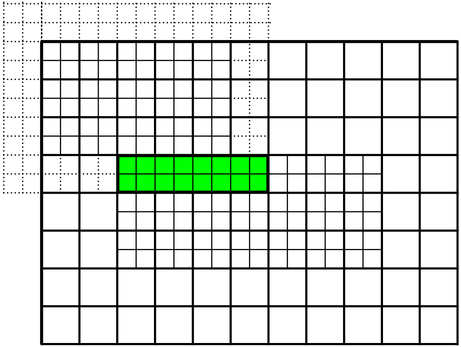

Synchronization with G_l - $\tilde{S}^s_{l,m} = \tilde{G}^s_{l,m} \cap G_l$

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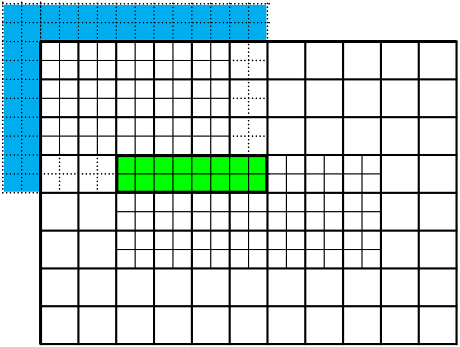

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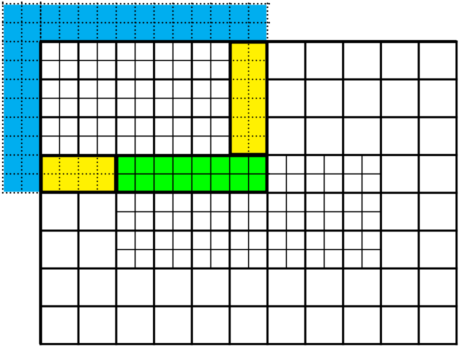

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Numerical update

Time-explicit conservative finite volume scheme

$$\mathcal{H}^{(\Delta t)}:\ \mathbf{Q}_{jk}(t+\Delta t) = \mathbf{Q}_{jk}(t) - \frac{\Delta t}{\Delta x_1}\left(\mathbf{F}^1_{j+\frac{1}{2},k} - \mathbf{F}^1_{j-\frac{1}{2},k}\right) - \frac{\Delta t}{\Delta x_2}\left(\mathbf{F}^2_{j,k+\frac{1}{2}} - \mathbf{F}^2_{j,k-\frac{1}{2}}\right)$$

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`UpdateLevel(`l`)`

`For all` $m = 1$ `To` M_l `Do`

$$\mathbf{Q}(G^s_{l,m}, t) \stackrel{\mathcal{H}^{(\Delta t_l)}}{\longrightarrow} \mathbf{Q}(G_{l,m}, t+\Delta t_l)\ , \mathbf{F}^n(\bar{G}_{l,m}, t)$$

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`If level` $l+1$ `exists`

`Init` $\delta\mathbf{F}^{n,l+1}$ `with` $\mathbf{F}^n(\bar{G}_{l,m} \cap \partial G_{l+1}, t)$

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If level $l > 0$

Add $\mathbf{F}^n(\partial G_{l,m}, t)$ to $\delta \mathbf{F}^{n,l}$

If level $l + 1$ exists

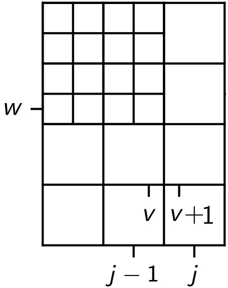
Init $\delta \mathbf{F}^{n,l+1}$ with $\mathbf{F}^n(\bar{G}_{l,m} \cap \partial G_{l+1}, t)$

Conservative flux correction

Example: Cell j, k

$$\check{\mathbf{Q}}^l_{jk}(t+\Delta t_l) = \mathbf{Q}^l_{jk}(t) - \frac{\Delta t_l}{\Delta x_{1,l}}\left(\mathbf{F}^{1,l}_{j+\frac{1}{2},k} - \frac{1}{r^2_{l+1}}\sum_{\kappa=0}^{r_{l+1}-1}\sum_{\iota=0}^{r_{l+1}-1}\mathbf{F}^{1,l+1}_{v+\frac{1}{2},w+\iota}(t+\kappa\Delta t_{l+1})\right)$$
$$- \frac{\Delta t_l}{\Delta x_{2,l}}\left(\mathbf{F}^{2,l}_{j,k+\frac{1}{2}} - \mathbf{F}^{2,l}_{j,k-\frac{1}{2}}\right)$$

Correction pass:



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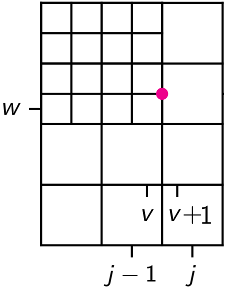

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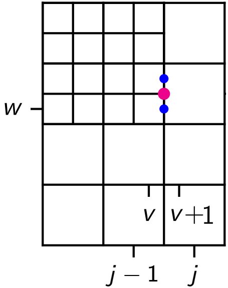

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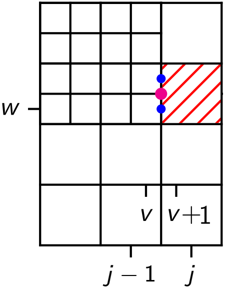
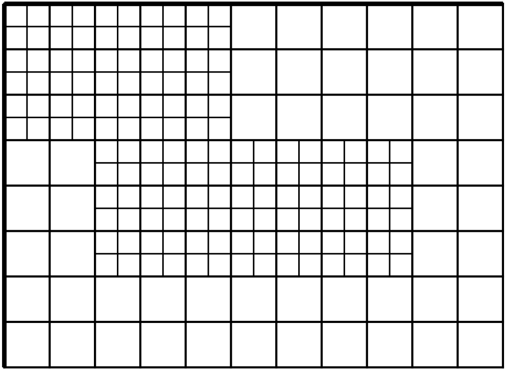
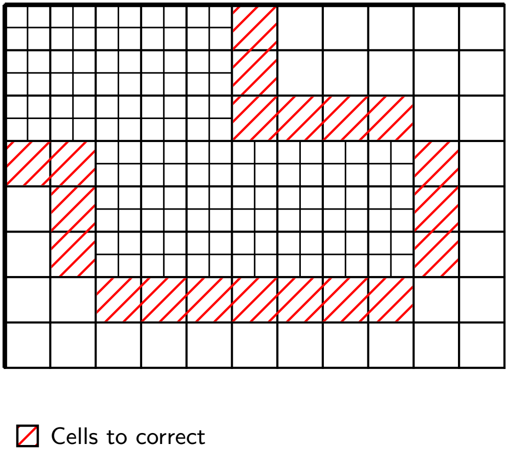


Conservative flux correction II



Conservative flux correction II

- Level l cells needing correction $(G_{l+1}^{r_{l+1}} \backslash G_{l+1}) \cap G_l$



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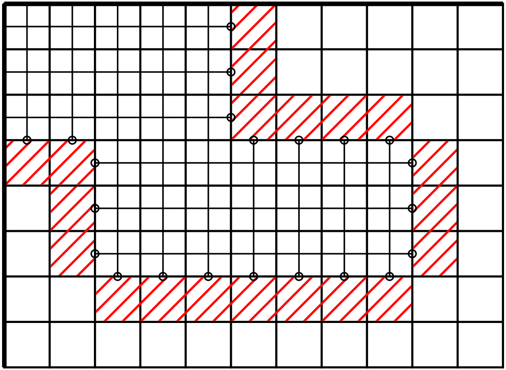
- Level l cells needing correction $(G_{l+1}^{r_{l+1}} \backslash G_{l+1}) \cap G_l$
- Corrections $\delta\mathbf{F}^{n,l+1}$ stored on level $l+1$ along ∂G_{l+1} (lower-dimensional data coarsened by r_{l+1})



Cells to correct ∘ $\delta\mathbf{F}^{n,l+1}$

Conservative flux correction II

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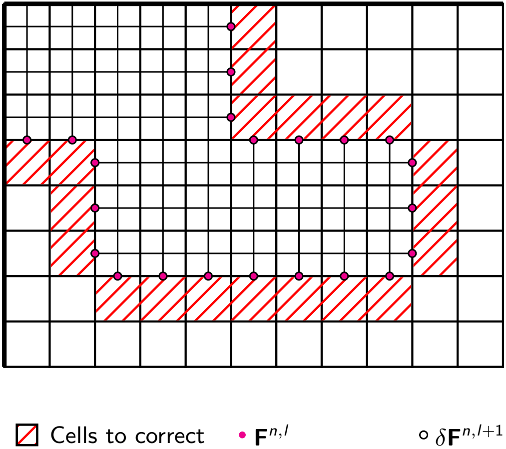

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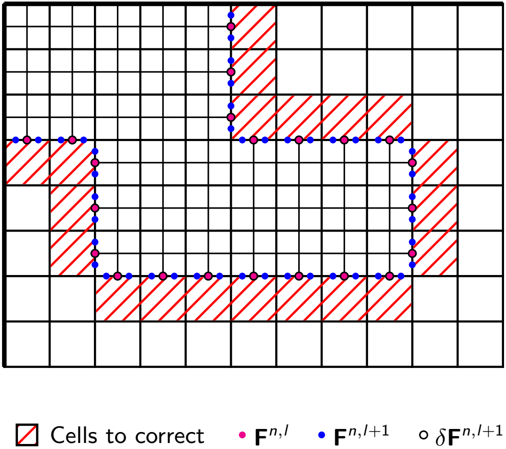
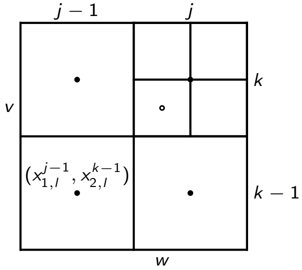
- Add level $l+1$ fluxes $\mathbf{F}^{n,l+1}(\partial G_{l+1})$ to $\delta \mathbf{F}^{n,l}$



Level transfer operators

Conservative averaging (restriction):
Replace cells on level l covered by level $l+1$, i.e. $G_l \cap G_{l+1}$, by

$$\hat{\mathbf{Q}}^l_{jk} := \frac{1}{(r_{l+1})^2} \sum_{\kappa=0}^{r_{l+1}-1} \sum_{\iota=0}^{r_{l+1}-1} \mathbf{Q}^{l+1}_{v+\kappa,w+\iota}$$



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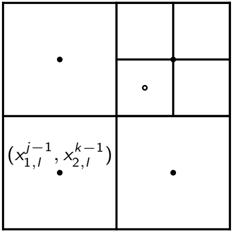

Bilinear interpolation (prolongation):

$$\check{\mathbf{Q}}^{l+1}_{vw} := (1-f_1)(1-f_2)\,\mathbf{Q}^l_{j-1,k-1} + f_1(1-f_2)\,\mathbf{Q}^l_{j,k-1} + (1-f_1)f_2\,\mathbf{Q}^l_{j-1,k} + f_1 f_2\,\mathbf{Q}^l_{jk}$$

with factors $f_1 := \dfrac{x^v_{1,l+1} - x^{j-1}_{1,l}}{\Delta x_{1,l}}$, $\quad f_2 := \dfrac{x^w_{2,l+1} - x^{k-1}_{2,l}}{\Delta x_{2,l}}$ derived from the spatial coordinates of the cell centers $(x^{j-1}_{1,l}, x^{k-1}_{2,l})$ and $(x^v_{1,l+1}, x^w_{2,l+1})$.

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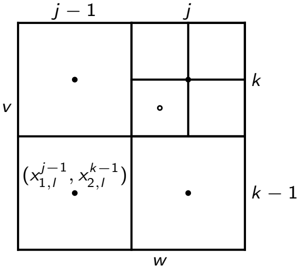

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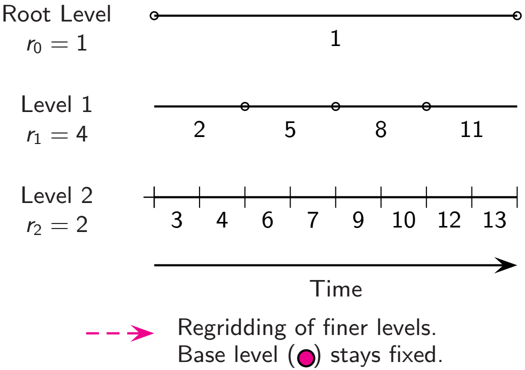
For boundary conditions on $\tilde{I}^s_l$: linear time interpolation

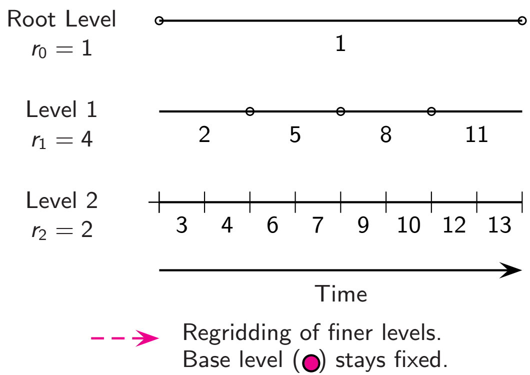
$$\tilde{\mathbf{Q}}^{l+1}(t+\kappa\Delta t_{l+1}) := \left(1 - \frac{\kappa}{r_{l+1}}\right)\check{\mathbf{Q}}^{l+1}(t) + \frac{\kappa}{r_{l+1}}\check{\mathbf{Q}}^{l+1}(t+\Delta t_l) \quad \text{for } \kappa = 0, \ldots r_{l+1}$$

Recursive integration order



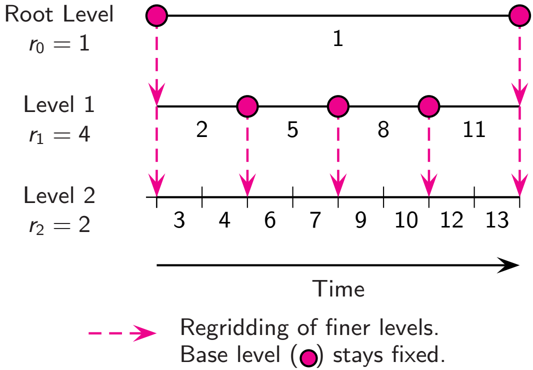
Recursive integration order

- Space-time interpolation of coarse data to set I_l^s, $l > 0$



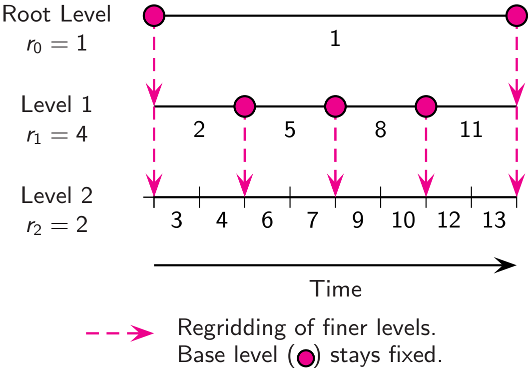
Recursive integration order

- Space-time interpolation of coarse data to set I_l^s, $l > 0$
- Regridding:
  - Creation of new grids, copy existing cells on level $l > 0$



Recursive integration order

- Space-time interpolation of coarse data to set I_l^s, $l > 0$
- Regridding:
  - Creation of new grids, copy existing cells on level $l > 0$
  - Spatial interpolation to initialize new cells on level $l > 0$



The basic recursive algorithm

```
AdvanceLevel(l)

  Repeat r_l times
      Set ghost cells of Q^l(t)


      UpdateLevel(l)





      t := t + Δt_l
```

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      If level l+1 exists?
          Set ghost cells of Q^l(t+Δt_l)
          AdvanceLevel(l+1)


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```

- Recursion

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          Average Q^{l+1}(t+Δt_l) onto Q^l(t+Δt_l)
          Correct Q^l(t+Δt_l) with δF^{l+1}
      t := t+Δt_l
```

- Recursion
- Restriction and flux correction

The basic recursive algorithm

```
AdvanceLevel(l)

  Repeat r_l times
      Set ghost cells of Q^l(t)
      If time to regrid?
          Regrid(l)
      UpdateLevel(l)
      If level l+1 exists?
          Set ghost cells of Q^l(t + Δt_l)
          AdvanceLevel(l+1)
          Average Q^(l+1)(t + Δt_l) onto Q^l(t + Δt_l)
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- Recursion
- Restriction and flux correction
- Re-organization of hierarchical data

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      t := t + Δt_l

Start - Start integration on level 0

     l = 0,  r_0 = 1
     AdvanceLevel(l)
```

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- Recursion
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[Berger and Colella, 1988][Berger and Oliger, 1984]

Regridding algorithm

Regrid(l) - Regrid all levels $\iota > l$

For $\iota = l_f$ Downto l Do
Flag N^ι according to $\mathbf{Q}^\iota(t)$

Regridding algorithm

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- Refinement flags: $N^l := \bigcup_m N(\partial G_{l,m})$

Regridding algorithm

```
Regrid(l) - Regrid all levels ι > l

  For ι = l_f Downto l Do
      Flag N^ι according to Q^ι(t)
      If level ι + 1 exists?
          Flag N^ι below Ğ^{ι+2}
```

- Refinement flags: $N^l := \bigcup_m N(\partial G_{l,m})$
- Activate flags below higher levels

Regridding algorithm

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Regrid(l) - Regrid all levels ι > l

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      If level ι+1 exists?
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      Flag buffer zone on N^ι
```

- Refinement flags: $N^l := \bigcup_m N(\partial G_{l,m})$
- Activate flags below higher levels
- Flag buffer cells of $b > \kappa_r$ cells, κ_r steps between calls of `Regrid(l)`

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Regridding algorithm

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      Generate Ğ^{ι+1} from N^ι
  Ğ_l := G_l
  For ι = l To l_f Do
      CĞ_ι := G_0 \ Ğ_ι
      Ğ_{ι+1} := Ğ_{ι+1} \ CĞ_ι^1
```

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- Use complement operation to ensure proper nesting condition

Regridding algorithm

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      Generate Ğ^{ι+1} from N^ι
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  For ι = l To l_f Do
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  Recompose(l)
```

- Refinement flags: $N^l := \bigcup_m N(\partial G_{l,m})$
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- Flag buffer cells of $b > \kappa_r$ cells, κ_r steps between calls of `Regrid(l)`
- Special cluster algorithm
- Use complement operation to ensure proper nesting condition

Recomposition of data

`Recompose`(l) `- Reorganize all levels` $\iota > l$

`For` $\iota = l+1$ `To` $l_f + 1$ `Do`

- Creates max. 1 level above l_f, but can remove multiple level if $\check{G}_\iota$ empty (no coarsening!)

Recomposition of data

Recompose(l) - Reorganize all levels $\iota > l$

For $\iota = l + 1$ To $l_f + 1$ Do
    Interpolate $\mathbf{Q}^{\iota-1}(t)$ onto $\check{\mathbf{Q}}^{\iota}(t)$

- Creates max. 1 level above l_f, but can remove multiple level if $\check{G}_\iota$ empty (no coarsening!)
- Use spatial interpolation on entire data $\check{\mathbf{Q}}^{\iota}(t)$

Recomposition of data

Recompose(l) - Reorganize all levels $\iota > l$

```
For ι = l + 1 To l_f + 1 Do
    Interpolate Q^{ι-1}(t) onto Q̌^ι(t)
    Copy Q^ι(t) onto Q̌^ι(t)
```

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- Use spatial interpolation on entire data $\check{\mathbf{Q}}^\iota(t)$
- Overwrite where old data exists

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Recompose(l) - Reorganize all levels ι > l

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      Copy Q^ι(t) onto Q̌^ι(t)
      Set ghost cells of Q̌^ι(t)
```

- Creates max. 1 level above l_f, but can remove multiple level if $\check{G}_\iota$ empty (no coarsening!)
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- Overwrite where old data exists
- Synchronization and physical boundary conditions

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      Interpolate Q^{ι−1}(t) onto Q̌^ι(t)
      Copy Q^ι(t) onto Q̌^ι(t)
      Set ghost cells of Q̌^ι(t)
      Q^ι(t) := Q̌^ι(t),  G_ι := Ǧ_ι
```

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Clustering by signatures

				x	x	x	x	x	x	6
				x	x	x	x	x	x	6
			x	x	x					3
	x	x	x							3
	x	x								2
	x	x								2
	x	x								2
										0
	x	x								2
	x	x								2
Υ	6	6	2	3	2	2	2	2	2	

Υ Flagged cells per row/column

Δ Second derivative of Υ, $\Delta = \Upsilon_{\nu+1} - 2\,\Upsilon_\nu + \Upsilon_{\nu-1}$

Technique from image detection: [Bell et al., 1994], see also [Berger and Rigoutsos, 1991], [Berger, 1986]

Clustering by signatures

				x	x	x	x	x	x	6
				x	x	x	x	x	x	6
			x	x	x					3
	x	x	x							3
	x	x								2
	x	x								2
	x	x								2
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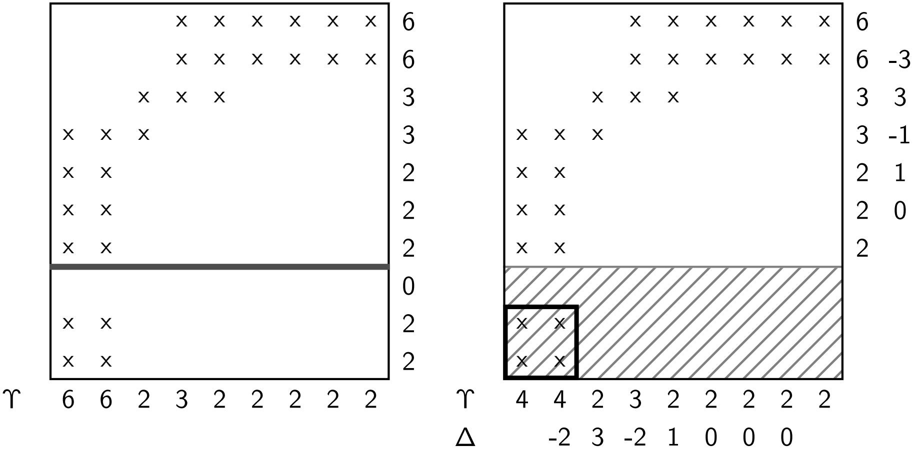

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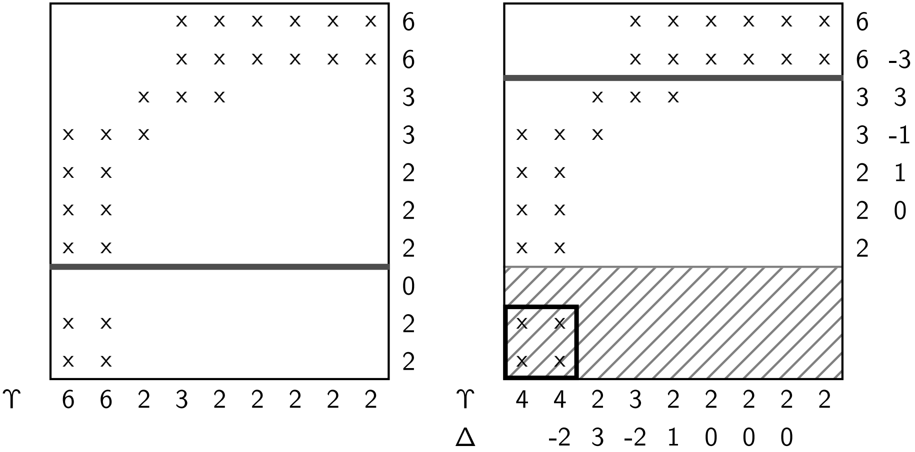

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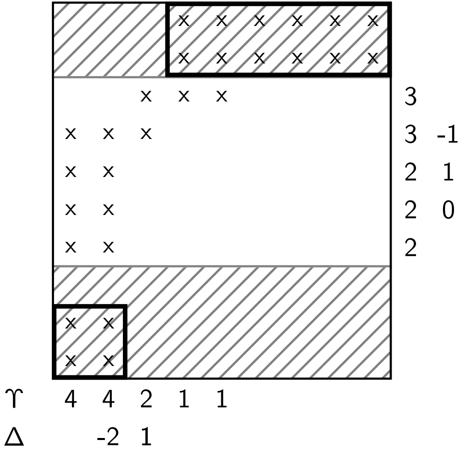
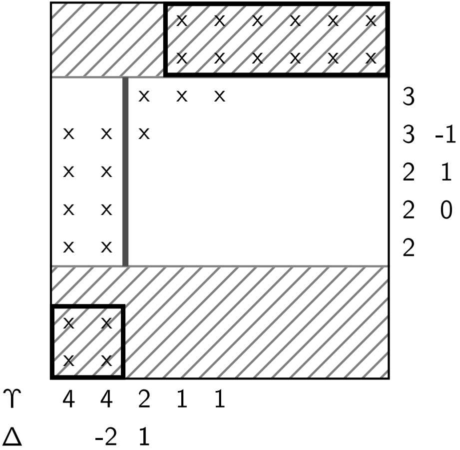


Recursive generation of $\breve{G}_{l,m}$

1. 0 in Υ
2. Largest difference in Δ
3. Stop if ratio between flagged and unflagged cell $> \eta_{tol}$



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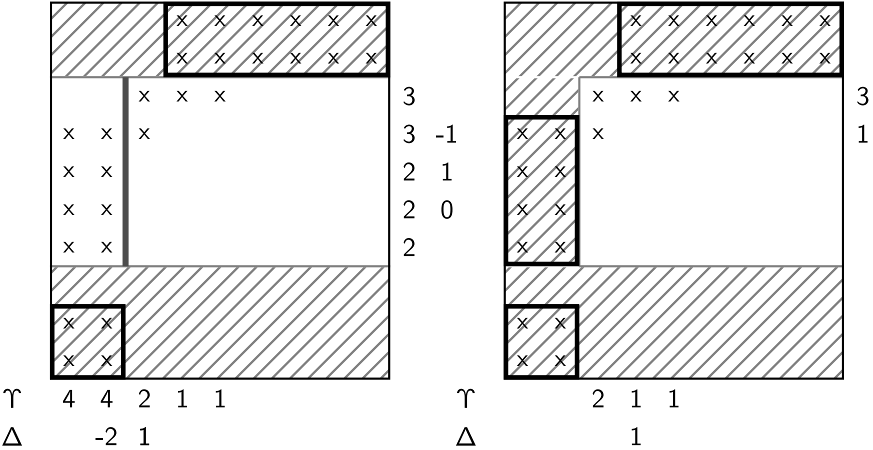

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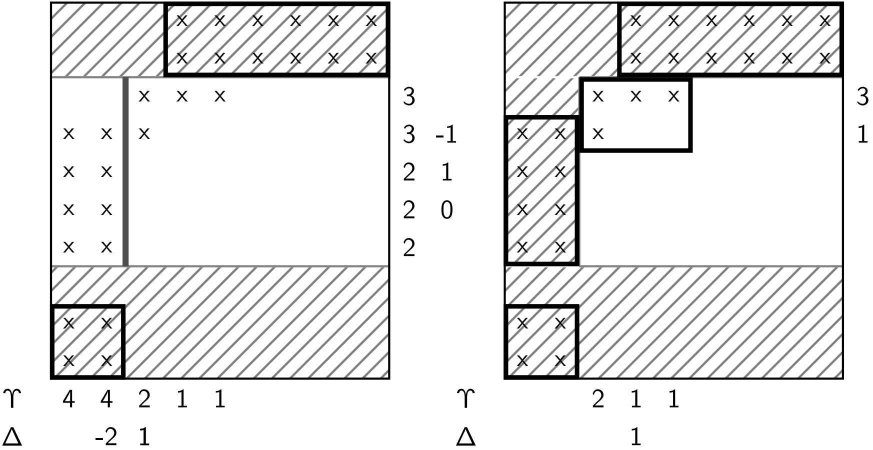

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Refinement criteria

Scaled gradient of scalar quantity w

$$|w(\mathbf{Q}_{j+1,k})-w(\mathbf{Q}_{jk})| > \epsilon_w\ ,\ |w(\mathbf{Q}_{j,k+1})-w(\mathbf{Q}_{jk})| > \epsilon_w\ ,\ |w(\mathbf{Q}_{j+1,k+1})-w(\mathbf{Q}_{jk})| > \epsilon_w$$

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Heuristic error estimation [Berger, 1982]:
Local truncation error of scheme of order o

$$\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}^{(\Delta t)}(\mathbf{q}(\cdot, t)) = \mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

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For $\mathbf{q}$ smooth after 2 steps Δt

$$\mathbf{q}(\mathbf{x}, t + \Delta t) - \mathcal{H}_2^{(\Delta t)}(\mathbf{q}(\cdot, t - \Delta t)) = 2\,\mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

Refinement criteria

Scaled gradient of scalar quantity w

$$|w(\mathbf{Q}_{j+1,k})-w(\mathbf{Q}_{jk})| > \epsilon_w \,,\; |w(\mathbf{Q}_{j,k+1})-w(\mathbf{Q}_{jk})| > \epsilon_w \,,\; |w(\mathbf{Q}_{j+1,k+1})-w(\mathbf{Q}_{jk})| > \epsilon_w$$

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For $\mathbf{q}$ smooth after 2 steps Δt

$$\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}_2^{(\Delta t)}(\mathbf{q}(\cdot, t-\Delta t)) = 2\,\mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

and after 1 step with $2\Delta t$

$$\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}^{(2\Delta t)}(\mathbf{q}(\cdot, t-\Delta t)) = 2^{o+1}\mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

Refinement criteria

Scaled gradient of scalar quantity w

$$|w(\mathbf{Q}_{j+1,k})-w(\mathbf{Q}_{jk})| > \epsilon_w\,,\ |w(\mathbf{Q}_{j,k+1})-w(\mathbf{Q}_{jk})| > \epsilon_w\,,\ |w(\mathbf{Q}_{j+1,k+1})-w(\mathbf{Q}_{jk})| > \epsilon_w$$

Heuristic error estimation [Berger, 1982]:
Local truncation error of scheme of order o

$$\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}^{(\Delta t)}(\mathbf{q}(\cdot, t)) = \mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

For $\mathbf{q}$ smooth after 2 steps Δt

$$\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}_2^{(\Delta t)}(\mathbf{q}(\cdot, t-\Delta t)) = 2\,\mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

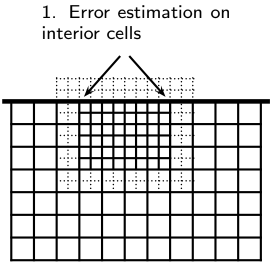
and after 1 step with $2\Delta t$

$$\mathbf{q}(\mathbf{x}, t+\Delta t) - \mathcal{H}^{(2\Delta t)}(\mathbf{q}(\cdot, t-\Delta t)) = 2^{o+1}\mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

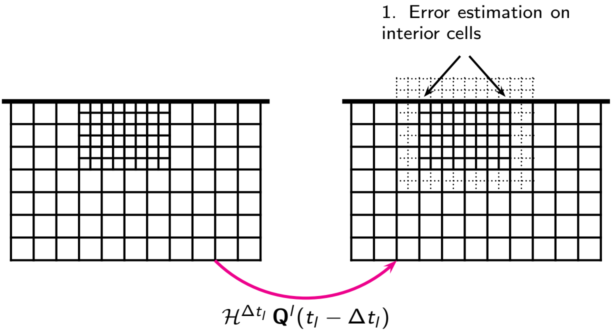
Gives

$$\mathcal{H}_2^{(\Delta t)}(\mathbf{q}(\cdot, t-\Delta t)) - \mathcal{H}^{(2\Delta t)}(\mathbf{q}(\cdot, t-\Delta t)) = (2^{o+1}-2)\mathbf{C}\Delta t^{o+1} + O(\Delta t^{o+2})$$

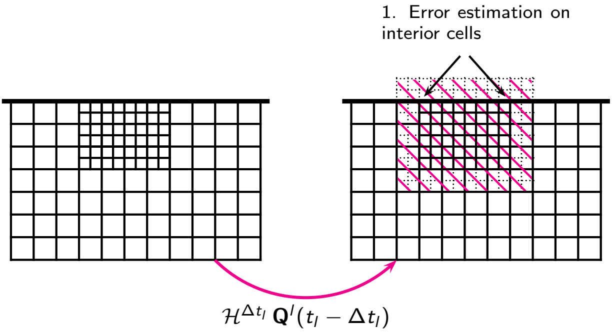
Heuristic error estimation for FV methods



Heuristic error estimation for FV methods



Heuristic error estimation for FV methods



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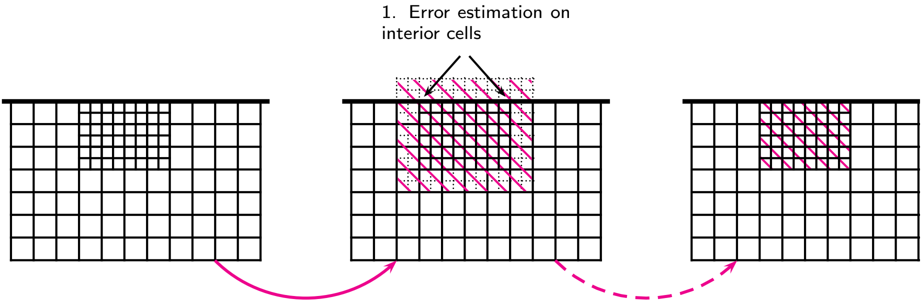


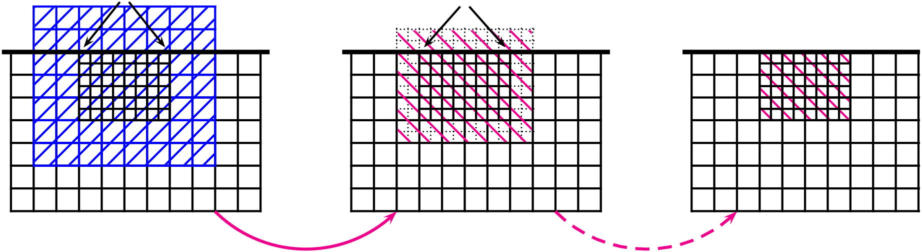
$$\mathcal{H}^{\Delta t_l}\,\mathbf{Q}^l(t_l-\Delta t_l)$$

$$\begin{aligned} & \mathcal{H}^{\Delta t_l}(\mathcal{H}^{\Delta t_l}\,\mathbf{Q}^l(t_l-\Delta t_l)) \\ = \;& \mathcal{H}_2^{\Delta t_l}\,\mathbf{Q}^l(t_l-\Delta t_l) \end{aligned}$$

Heuristic error estimation for FV methods

2. Create temporary Grid coarsened by factor 2
   Initialize with fine-grid-values of preceding time step

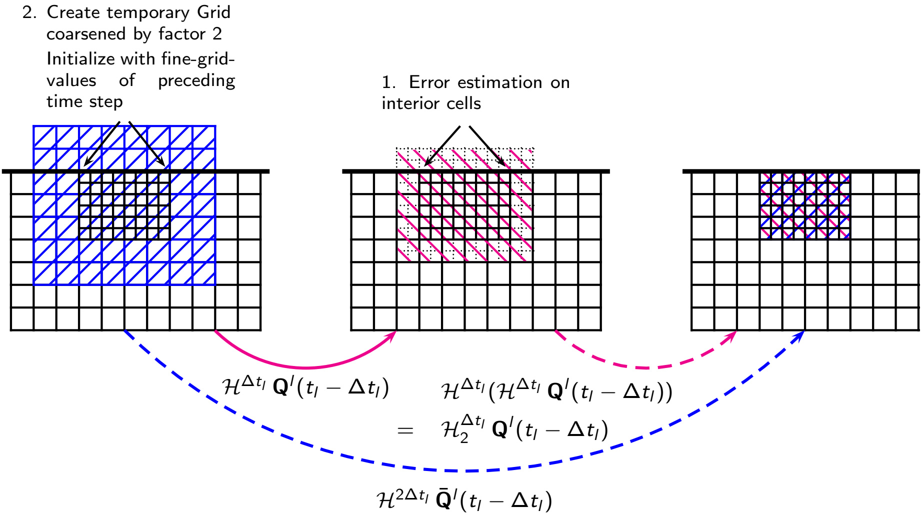
1. Error estimation on interior cells



$$\mathcal{H}^{\Delta t_l}\,\mathbf{Q}^l(t_l - \Delta t_l) \qquad \mathcal{H}^{\Delta t_l}(\mathcal{H}^{\Delta t_l}\,\mathbf{Q}^l(t_l - \Delta t_l))$$

$$= \quad \mathcal{H}_2^{\Delta t_l}\,\mathbf{Q}^l(t_l - \Delta t_l)$$

Heuristic error estimation for FV methods

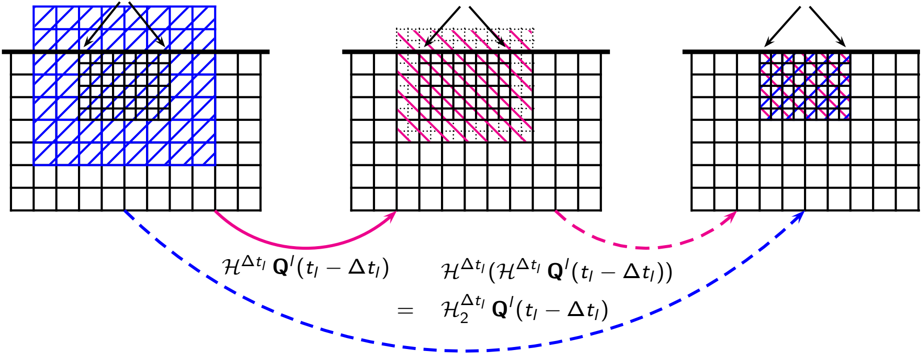


Heuristic error estimation for FV methods

2. Create temporary Grid coarsened by factor 2
   Initialize with fine-grid-values of preceding time step

1. Error estimation on interior cells

3. Compare temporary solutions



$$\mathcal{H}^{\Delta t_l}\,\mathbf{Q}^l(t_l-\Delta t_l)$$

$$\mathcal{H}^{\Delta t_l}(\mathcal{H}^{\Delta t_l}\,\mathbf{Q}^l(t_l-\Delta t_l)) = \mathcal{H}_2^{\Delta t_l}\,\mathbf{Q}^l(t_l-\Delta t_l)$$

$$\mathcal{H}^{2\Delta t_l}\,\bar{\mathbf{Q}}^l(t_l-\Delta t_l)$$

Usage of heuristic error estimation

Current solution integrated tentatively 1 step with Δt_l and coarsened

$$\bar{\mathcal{Q}}(t_l + \Delta t_l) := \text{Restrict}\left(\mathcal{H}_2^{\Delta t_l}\,\mathbf{Q}^l(t_l - \Delta t_l)\right)$$

Previous solution coarsened and integrated 1 step with $2\Delta t_l$

$$\mathcal{Q}(t_l + \Delta t_l) := \mathcal{H}^{2\Delta t_l}\,\text{Restrict}\left(\mathbf{Q}^l(t_l - \Delta t_l)\right)$$

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Local error estimation of scalar quantity w

$$\tau_{jk}^w := \frac{|w(\bar{\mathcal{Q}}_{jk}(t+\Delta t)) - w(\mathcal{Q}_{jk}(t+\Delta t))|}{2^{o+1} - 2}$$

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Current solution integrated tentatively 1 step with Δt_l and coarsened

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Local error estimation of scalar quantity w

$$\tau_{jk}^w := \frac{|w(\bar{\mathcal{Q}}_{jk}(t + \Delta t)) - w(\mathcal{Q}_{jk}(t + \Delta t))|}{2^{o+1} - 2}$$

In practice [Deiterding, 2003] use

$$\frac{\tau_{jk}^w}{\max(|w(\mathcal{Q}_{jk}(t + \Delta t))|, S_w)} > \eta_w^r$$

Outline

Meshes and adaptation
- Adaptivity on unstructured and structured meshes
- Available SAMR software

The serial Berger-Colella SAMR method
- Data structures and numerical update
- Conservative flux correction
- Level transfer operators
- The basic recursive algorithm
- Block generation and flagging of cells

Parallel SAMR method
- Domain decomposition
- A parallel SAMR algorithm

Parallelization strategies

Decomposition of the hierarchical data

- Distribution of each grid

Parallelization strategies

Decomposition of the hierarchical data

- Distribution of each grid
- Separate distribution of each level, cf. [Rendleman et al., 2000]

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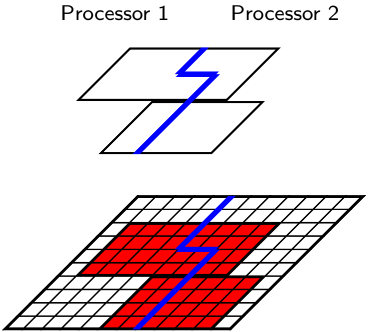
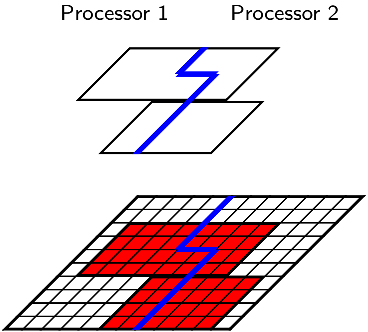
  - Data of all levels resides on same node



Parallelization strategies

Decomposition of the hierarchical data

- Distribution of each grid
- Separate distribution of each level, cf. [Rendleman et al., 2000]
- Rigorous domain decomposition
  - Data of all levels resides on same node
  - Grid hierarchy defines unique "floor-plan"



Parallelization strategies

Decomposition of the hierarchical data

- Distribution of each grid
- Separate distribution of each level, cf. [Rendleman et al., 2000]
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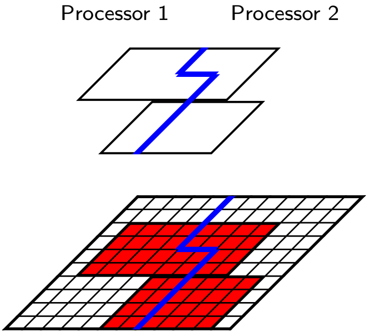
  - Grid hierarchy defines unique "floor-plan"
  - Redistribution of data blocks during reorganization of hierarchical data



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Decomposition of the hierarchical data

- Distribution of each grid
- Separate distribution of each level, cf. [Rendleman et al., 2000]
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  - Data of all levels resides on same node
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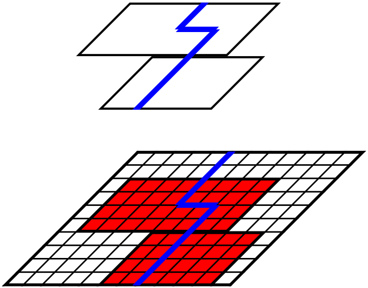
  - Redistribution of data blocks during reorganization of hierarchical data
  - Synchronization when setting ghost cells

Processor 1 Processor 2



Rigorous domain decomposition formalized

Parallel machine with P identical nodes. P non-overlapping portions G_0^p, $p = 1, \ldots, P$ as

$$G_0 = \bigcup_{p=1}^{P} G_0^p \quad \text{with} \quad G_0^p \cap G_0^q = \emptyset \;\; \text{for } p \neq q$$

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Higher level domains G_l follow decomposition of root level

$$G_l^p := G_l \cap G_0^p$$

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With $\mathcal{N}_l(\cdot)$ denoting number of cells, we estimate the workload as

$$\mathcal{W}(\Omega) = \sum_{l=0}^{l_{\max}} \left[\mathcal{N}_l(G_l \cap \Omega) \prod_{\kappa=0}^{l} r_\kappa \right]$$

Rigorous domain decomposition formalized

Parallel machine with P identical nodes. P non-overlapping portions G_0^p, $p = 1, \dots, P$ as

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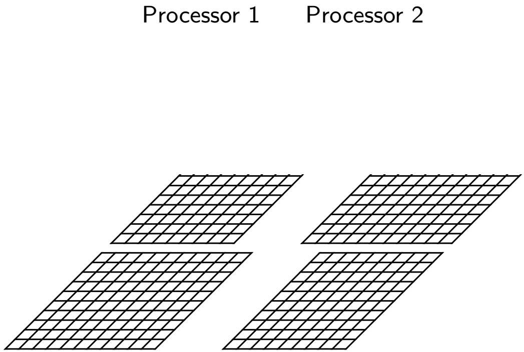
$$\mathcal{W}(\Omega) = \sum_{l=0}^{l_{\max}} \left[\mathcal{N}_l(G_l \cap \Omega) \prod_{\kappa=0}^{l} r_\kappa \right]$$

Equal work distribution necessitates

$$\mathcal{L}^p := \frac{P \cdot \mathcal{W}(G_0^p)}{\mathcal{W}(G_0)} \approx 1 \quad \text{for all } p = 1, \dots, P$$

[Deiterding, 2005]

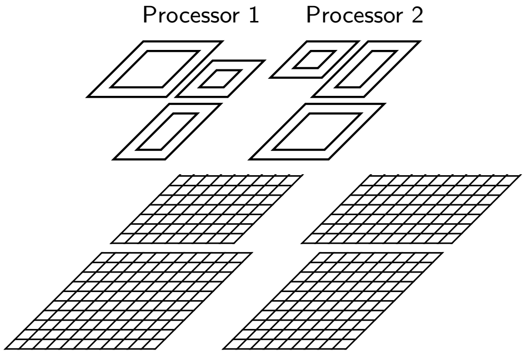
Ghost cell setting



Ghost cell values:

- Interpolation
- Local synchronization
- Parallel synchronization
- Physical boundary

Ghost cell setting



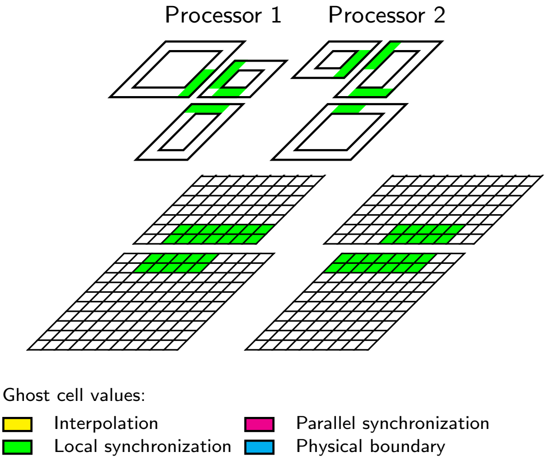
Ghost cell values:

(yellow)	Interpolation	(magenta)	Parallel synchronization
(green)	Local synchronization	(blue)	Physical boundary

Ghost cell setting

Local synchronization

$$\tilde{S}^{s,p}_{l,m} = \tilde{G}^{s,p}_{l,m} \cap G^p_l$$



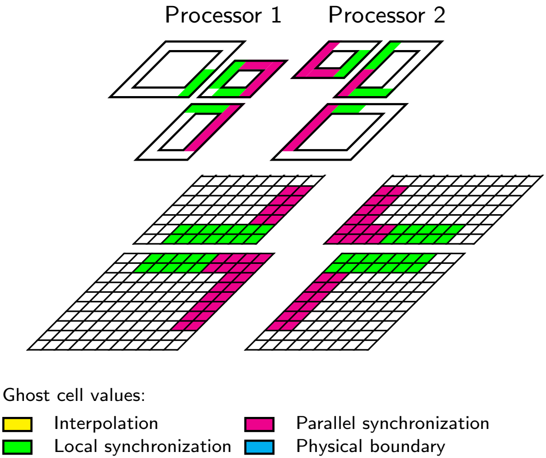
Ghost cell setting

Local synchronization

$$\tilde{S}^{s,p}_{l,m} = \tilde{G}^{s,p}_{l,m} \cap G^p_l$$

Parallel synchronization

$$\tilde{S}^{s,q}_{l,m} = \tilde{G}^{s,p}_{l,m} \cap G^q_l, q \neq p$$



Ghost cell setting

Local synchronization

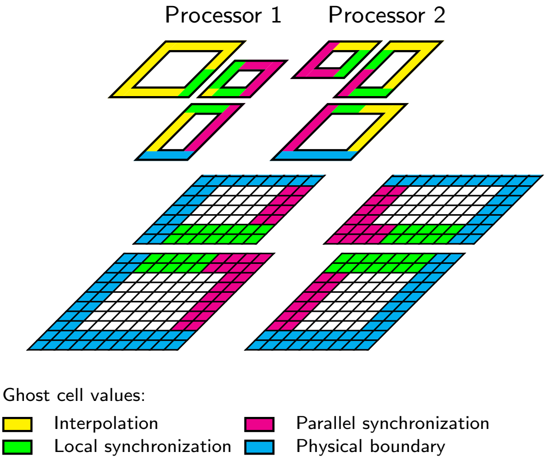
$$\tilde{S}^{s,p}_{l,m} = \tilde{G}^{s,p}_{l,m} \cap G^p_l$$

Parallel synchronization

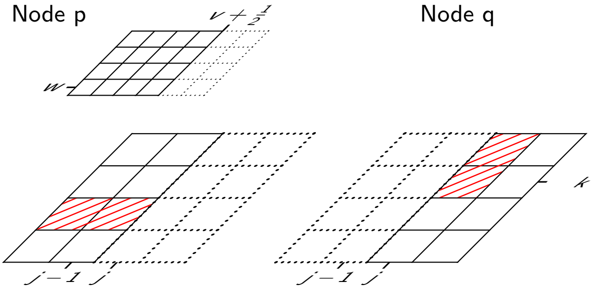
$$\tilde{S}^{s,q}_{l,m} = \tilde{G}^{s,p}_{l,m} \cap G^q_l, q \neq p$$

Interpolation and physical boundary conditions remain strictly local

- Scheme $\mathcal{H}^{(\Delta t_l)}$ evaluated locally
- Restriction and propolongation local

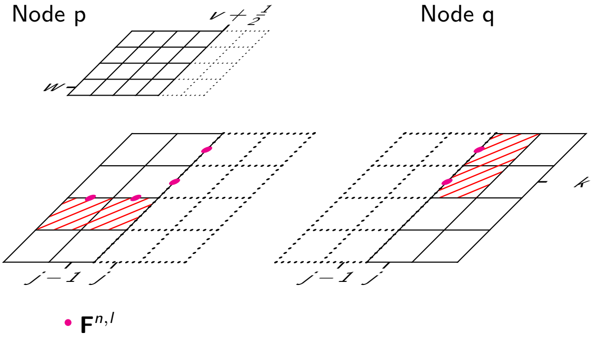


Parallel flux correction



Parallel flux correction

1. Strictly local: Init $\delta\mathbf{F}^{n,l+1}$ with $\mathbf{F}^n(\bar{G}_{l,m} \cap \partial G_{l+1}, t)$



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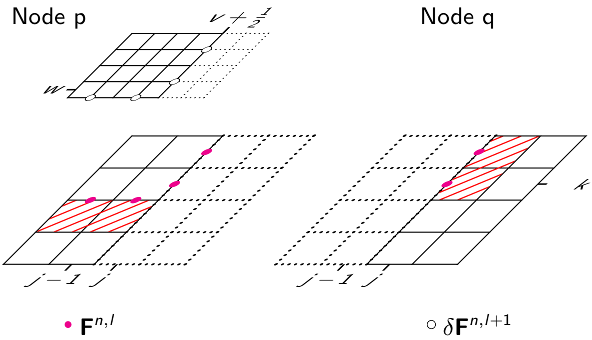
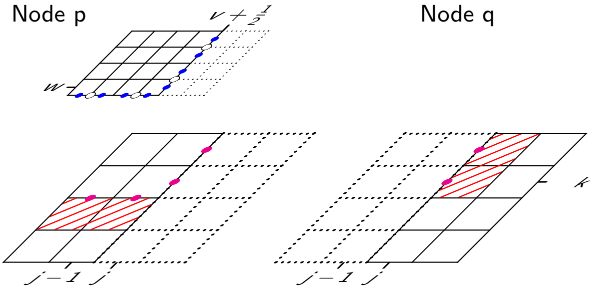
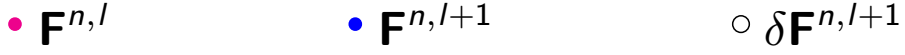


Parallel flux correction

1. Strictly local: Init $\delta\mathbf{F}^{n,l+1}$ with $\mathbf{F}^n(\bar{G}_{l,m} \cap \partial G_{l+1}, t)$
2. Strictly local: Add $\mathbf{F}^n(\partial G_{l,m}, t)$ to $\delta\mathbf{F}^{n,l}$

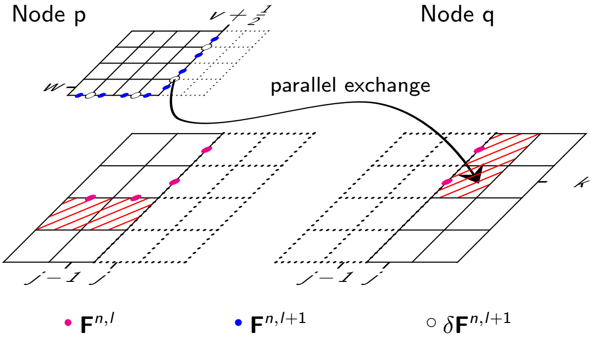




• $\mathbf{F}^{n,l}$ • $\mathbf{F}^{n,l+1}$ ○ $\delta\mathbf{F}^{n,l+1}$

Parallel flux correction

1. Strictly local: Init $\delta \mathbf{F}^{n,l+1}$ with $\mathbf{F}^n(\bar{G}_{l,m} \cap \partial G_{l+1}, t)$
2. Strictly local: Add $\mathbf{F}^n(\partial G_{l,m}, t)$ to $\delta \mathbf{F}^{n,l}$
3. Parallel communication: Correct $\mathbf{Q}^l(t + \Delta t_l)$ with $\delta \mathbf{F}^{l+1}$



The recursive algorithm in parallel

AdvanceLevel(l)

Repeat r_l times
  Set ghost cells of $\mathbf{Q}^l(t)$
  If time to regrid?
    Regrid(l)
  UpdateLevel(l)
  If level $l+1$ exists?
    Set ghost cells of $\mathbf{Q}^l(t+\Delta t_l)$
    AdvanceLevel($l+1$)
    Average $\mathbf{Q}^{l+1}(t+\Delta t_l)$ onto $\mathbf{Q}^l(t+\Delta t_l)$
    Correct $\mathbf{Q}^l(t+\Delta t_l)$ with $\delta\mathbf{F}^{l+1}$
  $t := t + \Delta t_l$

UpdateLevel(l)

For all $m = 1$ To M_l Do
  $\mathbf{Q}(G^s_{l,m}, t) \stackrel{\mathcal{H}^{(\Delta t_l)}}{\longrightarrow} \mathbf{Q}(G_{l,m}, t+\Delta t_l)\,, \mathbf{F}^n(\bar{G}_{l,m}, t)$
  If level $l > 0$
    Add $\mathbf{F}^n(\partial G_{l,m}, t)$ to $\delta\mathbf{F}^{n,l}$
  If level $l+1$ exists
    Init $\delta\mathbf{F}^{n,l+1}$ with $\mathbf{F}^n(\bar{G}_{l,m} \cap \partial G_{l+1}, t)$

The recursive algorithm in parallel

```
AdvanceLevel(l)

  Repeat r_l times
      Set ghost cells of Q^l(t)
      If time to regrid?
          Regrid(l)
      UpdateLevel(l)
      If level l+1 exists?
          Set ghost cells of Q^l(t+Δt_l)
          AdvanceLevel(l+1)
          Average Q^{l+1}(t+Δt_l) onto Q^l(t+Δt_l)
          Correct Q^l(t+Δt_l) with δF^{l+1}
      t := t+Δt_l

UpdateLevel(l)

  For all m = 1 To M_l Do
      Q(G^s_{l,m}, t) --H^(Δt_l)--> Q(G_{l,m}, t+Δt_l), F^n(Ḡ_{l,m}, t)
      If level l > 0
          Add F^n(∂G_{l,m}, t) to δF^{n,l}
      If level l+1 exists
          Init δF^{n,l+1} with F^n(Ḡ_{l,m} ∩ ∂G_{l+1}, t)
```

- Numerical update strictly local

The recursive algorithm in parallel

```
AdvanceLevel(l)

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      Set ghost cells of Q^l(t)
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          AdvanceLevel(l+1)
          Average Q^{l+1}(t+Δt_l) onto Q^l(t+Δt_l)
          Correct Q^l(t+Δt_l) with δF^{l+1}
      t := t+Δt_l

UpdateLevel(l)

  For all m = 1 To M_l Do
      Q(G^s_{l,m}, t) --H^(Δt_l)--> Q(G_{l,m}, t+Δt_l), F^n(Ḡ_{l,m}, t)
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```

- Numerical update strictly local
- Inter-level transfer local

The recursive algorithm in parallel

```
AdvanceLevel(l)

  Repeat r_l times
      Set ghost cells of Q^l(t)
      If time to regrid?
          Regrid(l)
      UpdateLevel(l)
      If level l+1 exists?
          Set ghost cells of Q^l(t+Δt_l)
          AdvanceLevel(l+1)
          Average Q^{l+1}(t+Δt_l) onto Q^l(t+Δt_l)
          Correct Q^l(t+Δt_l) with δF^{l+1}
      t := t+Δt_l

UpdateLevel(l)

  For all m = 1 To M_l Do
      Q(G^s_{l,m}, t) --H^(Δt_l)--> Q(G_{l,m}, t+Δt_l), F^n(Ḡ_{l,m}, t)
      If level l > 0
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      If level l+1 exists
          Init δF^{n,l+1} with F^n(Ḡ_{l,m} ∩ ∂G_{l+1}, t)
```

- Numerical update strictly local
- Inter-level transfer local
- Parallel synchronization

The recursive algorithm in parallel

```
AdvanceLevel(l)

  Repeat r_l times
      Set ghost cells of Q^l(t)
      If time to regrid?
          Regrid(l)
      UpdateLevel(l)
      If level l+1 exists?
          Set ghost cells of Q^l(t+Δt_l)
          AdvanceLevel(l+1)
          Average Q^{l+1}(t+Δt_l) onto Q^l(t+Δt_l)
          Correct Q^l(t+Δt_l) with δF^{l+1}
      t := t+Δt_l

UpdateLevel(l)

  For all m = 1 To M_l Do
      Q(G^s_{l,m}, t) --H^(Δt_l)--> Q(G_{l,m}, t+Δt_l) , F^n(Ḡ_{l,m}, t)
      If level l > 0
          Add F^n(∂G_{l,m}, t) to δF^{n,l}
      If level l+1 exists
          Init δF^{n,l+1} with F^n(Ḡ_{l,m} ∩ ∂G_{l+1}, t)
```

- Numerical update strictly local
- Inter-level transfer local
- Parallel synchronization
- Application of $\delta\mathbf{F}^{l+1}$ on ∂G_l^q

The recursive algorithm in parallel

```
AdvanceLevel(l)

  Repeat r_l times
      Set ghost cells of Q^l(t)
      If time to regrid?
          Regrid(l)
      UpdateLevel(l)
      If level l+1 exists?
          Set ghost cells of Q^l(t+Δt_l)
          AdvanceLevel(l+1)
          Average Q^{l+1}(t+Δt_l) onto Q^l(t+Δt_l)
          Correct Q^l(t+Δt_l) with δF^{l+1}
      t := t+Δt_l

UpdateLevel(l)

  For all m = 1 To M_l Do
      Q(G^s_{l,m}, t) --H^(Δt_l)--> Q(G_{l,m}, t+Δt_l), F^n(Ḡ_{l,m}, t)
      If level l > 0
          Add F^n(∂G_{l,m}, t) to δF^{n,l}
      If level l+1 exists
          Init δF^{n,l+1} with F^n(Ḡ_{l,m} ∩ ∂G_{l+1}, t)
```

- Numerical update strictly local
- Inter-level transfer local
- Parallel synchronization
- Application of $\delta\mathbf{F}^{l+1}$ on ∂G_l^q

Regridding algorithm in parallel

Regrid(l) - Regrid all levels $\iota > l$

For $\iota = l_f$ Downto l Do
    Flag N^ι according to $\mathbf{Q}^\iota(t)$
    If level $\iota + 1$ exists?
        Flag N^ι below $\breve{G}^{\iota+2}$
    Flag buffer zone on N^ι
    Generate $\breve{G}^{\iota+1}$ from N^ι
$\breve{G}_l := G_l$
For $\iota = l$ To l_f Do
    $C\breve{G}_\iota := G_0 \backslash \breve{G}_\iota$
    $\breve{G}_{\iota+1} := \breve{G}_{\iota+1} \backslash C\breve{G}_\iota^1$

Recompose(l)

Regridding algorithm in parallel

```
Regrid(l) - Regrid all levels ι > l

  For ι = l_f Downto l Do
      Flag N^ι according to Q^ι(t)
      If level ι+1 exists?
          Flag N^ι below Ğ^{ι+2}
      Flag buffer zone on N^ι
      Generate Ğ^{ι+1} from N^ι
  Ğ_l := G_l
  For ι = l To l_f Do
      CĞ_ι := G_0 \ Ğ_ι
      Ğ_{ι+1} := Ğ_{ι+1} \ CĞ_ι^1

  Recompose(l)
```

Regridding algorithm in parallel

```
Regrid(l) - Regrid all levels ι > l

  For ι = l_f Downto l Do
      Flag N^ι according to Q^ι(t)
      If level ι+1 exists?
          Flag N^ι below Ğ^(ι+2)
      Flag buffer zone on N^ι
      Generate Ğ^(ι+1) from N^ι
  Ğ_l := G_l
  For ι = l To l_f Do
      CĞ_ι := G_0 \ Ğ_ι
      Ğ_(ι+1) := Ğ_(ι+1) \ CĞ_ι^1

  Recompose(l)
```

- Need a ghost cell overlap of b cells to ensure correct setting of refinement flags in parallel

Regridding algorithm in parallel

```
Regrid(l) - Regrid all levels ι > l

  For ι = l_f Downto l Do
      Flag N^ι according to Q^ι(t)
      If level ι+1 exists?
          Flag N^ι below Ğ^{ι+2}
      Flag buffer zone on N^ι
      Generate Ğ^{ι+1} from N^ι
  Ğ_l := G_l
  For ι = l To l_f Do
      CĞ_ι := G_0 \ Ğ_ι
      Ğ_{ι+1} := Ğ_{ι+1} \ CĞ_ι^1

  Recompose(l)
```

- Need a ghost cell overlap of b cells to ensure correct setting of refinement flags in parallel
- Two options exist (we choose the latter):
  - Global clustering algorithm
  - Local clustering algorithm and concatenation of new lists $\breve{G}^{\iota+1}$

Regridding algorithm in parallel

`Regrid(`l`) - Regrid all levels` $\iota > l$

For $\iota = l_f$ Downto l Do
  Flag N^ι according to $\mathbf{Q}^\iota(t)$
  If level $\iota + 1$ exists?
    Flag N^ι below $\breve{G}^{\iota+2}$
  Flag buffer zone on N^ι
  Generate $\breve{G}^{\iota+1}$ from N^ι
$\breve{G}_l := G_l$
For $\iota = l$ To l_f Do
  $C\breve{G}_\iota := G_0 \setminus \breve{G}_\iota$
  $\breve{G}_{\iota+1} := \breve{G}_{\iota+1} \setminus C\breve{G}_\iota^1$
Recompose(l)

- Need a ghost cell overlap of b cells to ensure correct setting of refinement flags in parallel
- Two options exist (we choose the latter):
  - Global clustering algorithm
  - Local clustering algorithm and concatenation of new lists $\breve{G}^{\iota+1}$

Regridding algorithm in parallel

Regrid(l) - Regrid all levels $\iota > l$

For $\iota = l_f$ Downto l Do
  Flag N^ι according to $\mathbf{Q}^\iota(t)$
  If level $\iota + 1$ exists?
    Flag N^ι below $\breve{G}^{\iota+2}$
  Flag buffer zone on N^ι
  Generate $\breve{G}^{\iota+1}$ from N^ι
$\breve{G}_l := G_l$
For $\iota = l$ To l_f Do
  $C\breve{G}_\iota := G_0 \setminus \breve{G}_\iota$
  $\breve{G}_{\iota+1} := \breve{G}_{\iota+1} \setminus C\breve{G}_\iota^1$

Recompose(l)

- Need a ghost cell overlap of b cells to ensure correct setting of refinement flags in parallel
- Two options exist (we choose the latter):
  - Global clustering algorithm
  - Local clustering algorithm and concatenation of new lists $\breve{G}^{\iota+1}$

Recomposition algorithm in parallel

`Recompose(`l`) - Reorganize all levels`

`For` $\iota = l+1$ `To` l_f+1 `Do`

`Interpolate` $\mathbf{Q}^{\iota-1}(t)$ `onto` $\check{\mathbf{Q}}^{\iota}(t)$

`Copy` $\mathbf{Q}^{\iota}(t)$ `onto` $\check{\mathbf{Q}}^{\iota}(t)$
`Set ghost cells of` $\check{\mathbf{Q}}^{\iota}(t)$
$\mathbf{Q}^{\iota}(t) := \check{\mathbf{Q}}^{\iota}(t)$
$G_\iota := \check{G}_\iota$

Recomposition algorithm in parallel

Recompose(l) - Reorganize all levels

Generate G_0^p from $\{G_0, ..., G_l, \breve{G}_{l+1}, ..., \breve{G}_{l_f+1}\}$
For $\iota = 0$ To $l_f + 1$ Do

Interpolate $\mathbf{Q}^{\iota-1}(t)$ onto $\breve{\mathbf{Q}}^{\iota}(t)$

Copy $\mathbf{Q}^{\iota}(t)$ onto $\breve{\mathbf{Q}}^{\iota}(t)$
Set ghost cells of $\breve{\mathbf{Q}}^{\iota}(t)$
$\mathbf{Q}^{\iota}(t) := \breve{\mathbf{Q}}^{\iota}(t)$
$G_\iota^p := \breve{G}_\iota^p$, $G_\iota := \bigcup_p G_\iota^p$

- Global redistribution can also be required when regridding higher levels and $G_0, ..., G_l$ do not change (drawback of domain decomposition)

Recomposition algorithm in parallel

Recompose(l) - Reorganize all levels

Generate G_0^p from $\{G_0, ..., G_l, \breve{G}_{l+1}, ..., \breve{G}_{l_f+1}\}$
For $\iota = 0$ To $l_f + 1$ Do
  If $\iota > l$
    $\breve{G}_\iota^p := \breve{G}_\iota \cap G_0^p$
    Interpolate $\mathbf{Q}^{\iota-1}(t)$ onto $\breve{\mathbf{Q}}^\iota(t)$

    Copy $\mathbf{Q}^\iota(t)$ onto $\breve{\mathbf{Q}}^\iota(t)$
    Set ghost cells of $\breve{\mathbf{Q}}^\iota(t)$
    $\mathbf{Q}^\iota(t) := \breve{\mathbf{Q}}^\iota(t)$
  $G_\iota^p := \breve{G}_\iota^p$, $G_\iota := \bigcup_p G_\iota^p$

- Global redistribution can also be required when regridding higher levels and $G_0, ..., G_l$ do not change (drawback of domain decomposition)
- When $\iota > l$ do nothing special
- For $\iota \le l$, redistribute additionally

Recomposition algorithm in parallel

```
Recompose(l) - Reorganize all levels

   Generate G_0^p from {G_0, ..., G_l, Ğ_{l+1}, ..., Ğ_{l_f+1}}
   For ι = 0 To l_f + 1 Do
        If ι > l
             Ğ_ι^p := Ğ_ι ∩ G_0^p
             Interpolate Q^{ι-1}(t) onto Q̌^ι(t)
        else
             Ğ_ι^p := G_ι ∩ G_0^p
             If ι > 0
                  Copy δF^{n,ι} onto δF̌^{n,ι}
                  δF^{n,ι} := δF̌^{n,ι}

             Copy Q^ι(t) onto Q̌^ι(t)
             Set ghost cells of Q̌^ι(t)
             Q^ι(t) := Q̌^ι(t)
        G_ι^p := Ğ_ι^p ,  G_ι := ⋃_p G_ι^p
```

- Global redistribution can also be required when regridding higher levels and $G_0, \ldots, G_l$ do not change (drawback of domain decomposition)
- When $\iota > l$ do nothing special
- For $\iota \le l$, redistribute additionally
    - Flux corrections $\delta\mathbf{F}^{n,\iota}$

Recomposition algorithm in parallel

```
Recompose(l) - Reorganize all levels

    Generate G_0^p from {G_0, ..., G_l, Ğ_{l+1}, ..., Ğ_{l_f+1}}
    For ι = 0 To l_f + 1 Do
        If ι > l
            Ğ_ι^p := Ğ_ι ∩ G_0^p
            Interpolate Q^{ι-1}(t) onto Q̌^ι(t)
        else
            Ğ_ι^p := G_ι ∩ G_0^p
            If ι > 0
                Copy δF^{n,ι} onto δF̌^{n,ι}
                δF^{n,ι} := δF̌^{n,ι}
        If ι ≥ l then κ_ι = 0 else κ_ι = 1
        For κ = 0 To κ_ι Do
            Copy Q^ι(t + κΔt_ι) onto Q̌^ι(t + κΔt_ι)
            Set ghost cells of Q̌^ι(t + κΔt_ι)
            Q^ι(t + κΔt_ι) := Q̌^ι(t + κΔt_ι)
        G_ι^p := Ğ_ι^p,  G_ι := ⋃_p G_ι^p
```

- Global redistribution can also be required when regridding higher levels and $G_0, \ldots, G_l$ do not change (drawback of domain decomposition)
- When $\iota > l$ do nothing special
- For $\iota \le l$, redistribute additionally
  - Flux corrections $\delta\mathbf{F}^{n,\iota}$
  - Already updated time level $\mathbf{Q}^{\iota}(t + \kappa\Delta t_\iota)$

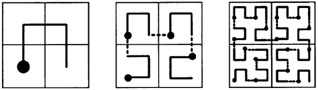
Recomposition algorithm in parallel

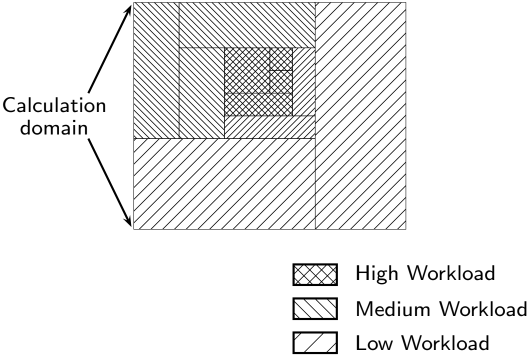
```
Recompose(l) - Reorganize all levels

    Generate G_0^p from {G_0, ..., G_l, Ğ_{l+1}, ..., Ğ_{l_f+1}}
    For ι = 0 To l_f + 1 Do
        If ι > l
            Ğ_ι^p := Ğ_ι ∩ G_0^p
            Interpolate Q^{ι-1}(t) onto Q̌^ι(t)
        else
            Ğ_ι^p := G_ι ∩ G_0^p
            If ι > 0
                Copy δF^{n,ι} onto δF̌^{n,ι}
                δF^{n,ι} := δF̌^{n,ι}
        If ι ≥ l then κ_ι = 0 else κ_ι = 1
        For κ = 0 To κ_ι Do
            Copy Q^ι(t + κΔt_ι) onto Q̌^ι(t + κΔt_ι)
            Set ghost cells of Q̌^ι(t + κΔt_ι)
            Q^ι(t + κΔt_ι) := Q̌^ι(t + κΔt_ι)
        G_ι^p := Ğ_ι^p ,  G_ι := ⋃_p G_ι^p
```

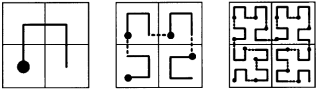
- Global redistribution can also be required when regridding higher levels and $G_0, \ldots, G_l$ do not change (drawback of domain decomposition)
- When $\iota > l$ do nothing special
- For $\iota \le l$, redistribute additionally
  - Flux corrections $\delta\mathbf{F}^{n,\iota}$
  - Already updated time level $\mathbf{Q}^{\iota}(t+\kappa\Delta t_\iota)$

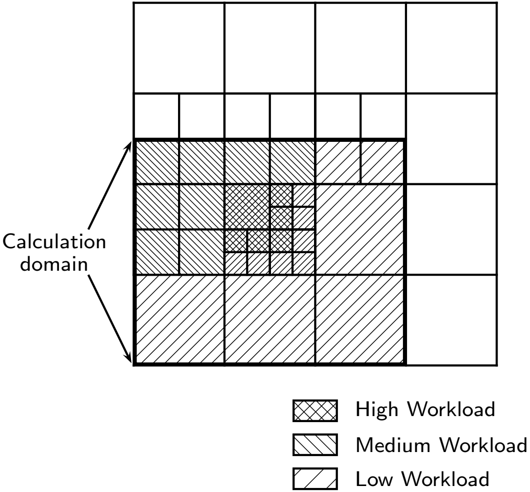
Space-filling curve algorithm



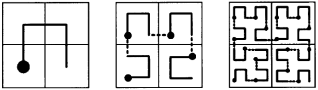


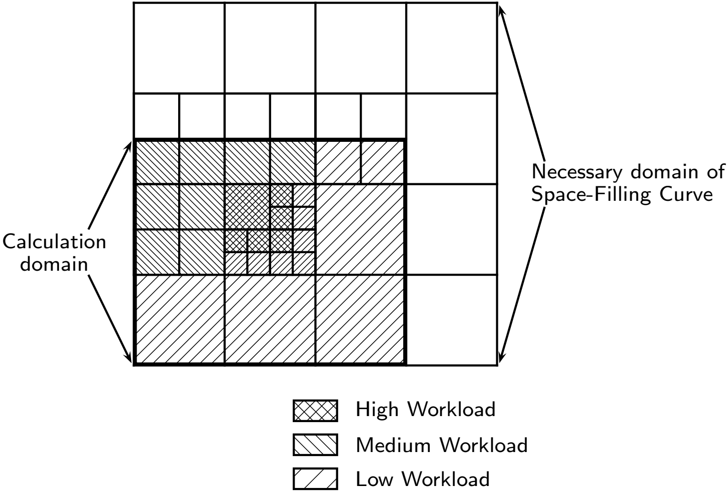
Space-filling curve algorithm



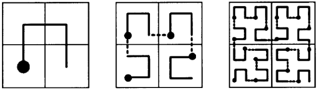


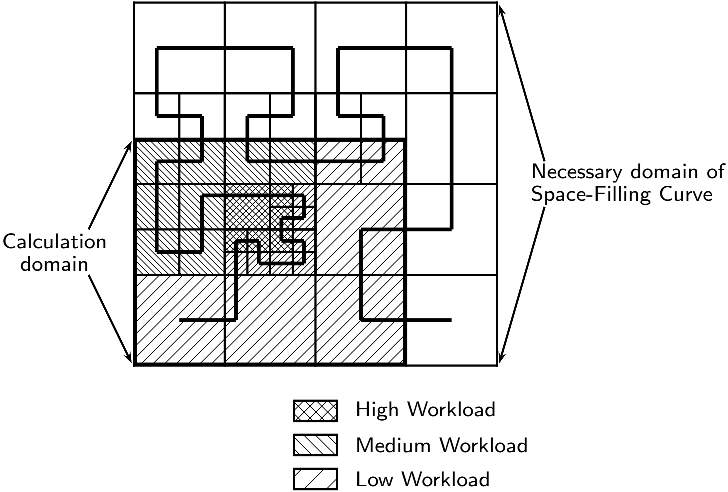
Space-filling curve algorithm



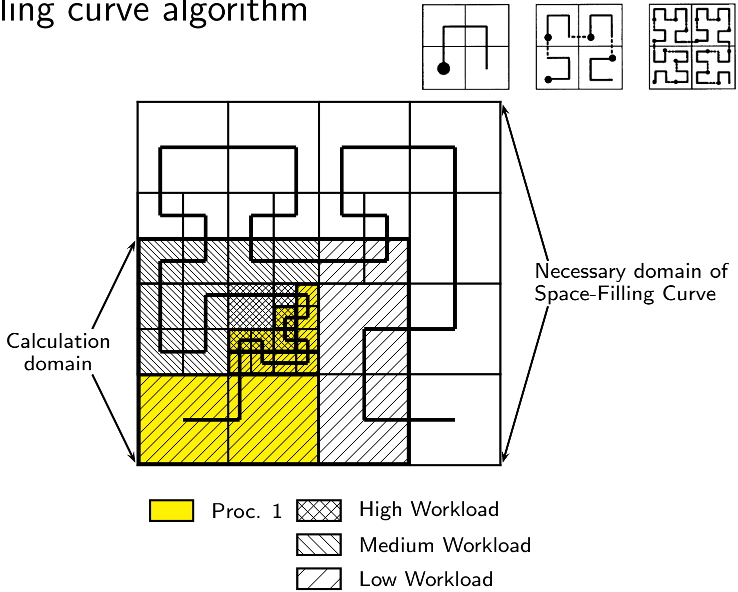


Space-filling curve algorithm

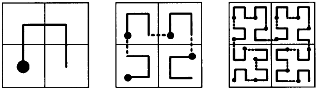


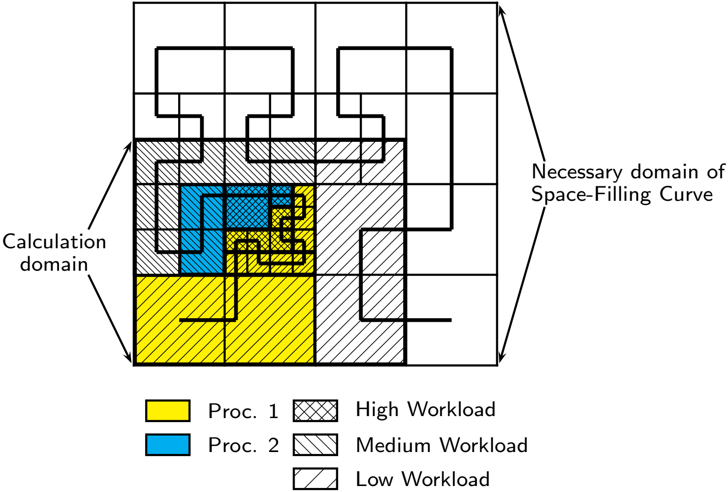


Space-filling curve algorithm

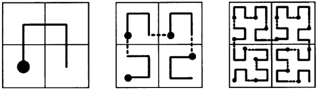


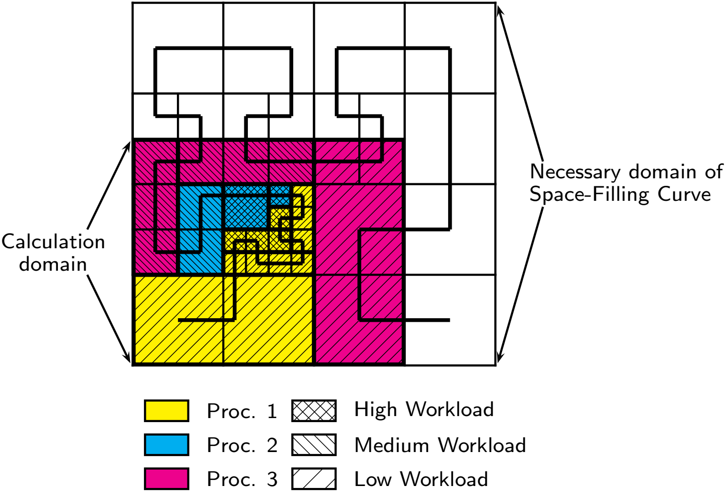
Space-filling curve algorithm





Space-filling curve algorithm





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