Design and application of wavelet-based refinement criteria for hyperbolic conservation laws within the AMROC mesh adaptation framework

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Outline

Principles of SAMR
- Block-structured adaptive mesh refinement
- Common refinement criteria
- Implementation in AMROC

Multiresolution techniques
- Multiresolution principles
- New MR refinement criteria

Computational results for Euler equations
- Verification
- Lax–Liu test cases

Computational results for magneto-hydrodynamics
- Ideal magneto-hydrodynamics simulation

Conclusions
- Summary and outlook
Collaboration with

- Kai Schneider and Oliver Roussel (University of Marseille, France)
- Muller Moreira Lopes (INPE – LAC/CTE)
Block-structured adaptive mesh refinement (SAMR)

For simplicity $\partial_t q(x, t) + \nabla \cdot f(q(x, t)) = 0$

- Refined blocks overlay coarser ones
Block-structured adaptive mesh refinement (SAMR)

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- Refined blocks overlay coarser ones
- Refinement in space and time by factor $r_t$
- Block (aka patch) based data structures
  + Numerical scheme

$$Q_{jk}^{n+1} = Q_{jk}^n - \frac{\Delta t}{\Delta x} \left[ F_{j+\frac{1}{2}, k} - F_{j-\frac{1}{2}, k} \right]$$

$$- \frac{\Delta t}{\Delta y} \left[ G_{j, k+\frac{1}{2}} - G_{j, k-\frac{1}{2}} \right]$$

only for single patch necessary

+ Efficient cache-reuse / vectorization possible
  - Cluster-algorithm necessary
Level transfer / setting of ghost cells

Conservative averaging (restriction):

\[ Q_{jk}^{l} := \frac{1}{(r_{l+1})^2} \left( \sum_{\kappa=0}^{r_{l+1}-1} \sum_{\iota=0}^{r_{l+1}-1} Q_{v+\kappa, w+\iota}^{l+1} \right) \]
Level transfer / setting of ghost cells

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Bilinear interpolation (prolongation):

\[ \check{Q}_{v+w}^{l+1} := (1 - f_1)(1 - f_2) Q_{j-1, k-1}^l + f_1 (1 - f_2) Q_{j, k-1}^l + (1 - f_1) f_2 Q_{j-1, k}^l + f_1 f_2 Q_{jk}^l \]

For boundary conditions: linear time interpolation

\[ \tilde{Q}_{v+w}^{l+1}(t + \kappa \Delta t_{l+1}) := \left(1 - \frac{\kappa}{r_{l+1}}\right) \check{Q}_{v+w}^{l+1}(t) + \frac{\kappa}{r_{l+1}} \check{Q}_{v+w}^{l+1}(t + \Delta t_l) \text{ for } \kappa = 0, \ldots, r_{l+1} \]
Conservative flux correction

Example: Cell \( j, k \)

\[
\dot{Q}_{jk}^l(t + \Delta t_l) = Q_{jk}^l(t) - \frac{\Delta t_l}{\Delta x_l} \left( F^{1,l}_{j+k+1/2,k} - \frac{1}{r_{l+1}^2} \sum_{\kappa=0}^{r_{l+1}-1} \sum_{\iota=0}^{r_{l+1}-1} F^{1,l+1}_{v+w+1/2,\nu+1/2,\omega+\iota}(t + \kappa \Delta t_{l+1}) \right) \\
- \frac{\Delta t_l}{\Delta y_l} \left( F^{2,l}_{j,k+1/2} - F^{2,l}_{j,k-1/2} \right)
\]

Correction pass:
Conservative flux correction

Example: Cell $j, k$

\[
\hat{Q}'_{jk}(t + \Delta t_l) = Q'_{jk}(t) - \frac{\Delta t_l}{\Delta x_l} \left( F_{j+\frac{1}{2},k}^{1,l} - \frac{1}{r_{l+1}^2} \sum_{\kappa=0}^{r_{l+1}-1} \sum_{\iota=0}^{r_{l+1}-1} F_{v+\frac{1}{2},w+\iota}^{1,l+1} (t + \kappa \Delta t_{l+1}) \right) \\
- \frac{\Delta t_l}{\Delta y_l} \left( F_{j,k+\frac{1}{2}}^{2,l} - F_{j,k-\frac{1}{2}}^{2,l} \right)
\]

Correction pass:

1. $\delta F_{j-\frac{1}{2},k}^{1,l+1} := -F_{j-\frac{1}{2},k}^{1,l}$

\[
\hat{Q}'_{jk}(t + \Delta t_l) = Q'_{jk}(t) - \frac{\Delta t_l}{\Delta x_l} \left( F_{j+\frac{1}{2},k}^{1,l} - \frac{1}{r_{l+1}^2} \sum_{\kappa=0}^{r_{l+1}-1} \sum_{\iota=0}^{r_{l+1}-1} F_{v+\frac{1}{2},w+\iota}^{1,l+1} (t + \kappa \Delta t_{l+1}) \right) \\
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- \frac{\Delta t_l}{\Delta y_l} \left( F_{j,k+\frac{1}{2}}^{2,l} - F_{j,k-\frac{1}{2}}^{2,l} \right)
\]
Conservative flux correction

Example: Cell \( j, k \)

\[
\dot{Q}_{jk}^l(t + \Delta t_l) = Q_{jk}^l(t) - \frac{\Delta t_l}{\Delta x_l} \left( F_1^{1,l}, j + \frac{1}{2}, k - \frac{1}{r_{l+1}} \sum_{\kappa=0}^{r_{l+1}-1} \sum_{\iota=0}^{1} F_1^{1,l+1}, v + \frac{1}{2}, w + \iota (t + \kappa \Delta t_{l+1}) \right)
\]

\[
- \frac{\Delta t_l}{\Delta y_l} \left( F_2^{2,l}, j, k + \frac{1}{2} - F_2^{2,l}, j, k - \frac{1}{2} \right)
\]

Correction pass:

1. \( \delta F_1^{1,l+1}, j - \frac{1}{2}, k := -F_1^{1,l}, j - \frac{1}{2}, k \)

2. \( \delta F_1^{1,l+1}, j - \frac{1}{2}, k := \delta F_1^{1,l+1}, j - \frac{1}{2}, k + \frac{1}{r_{l+1}} \sum_{\iota=0}^{1} F_1^{1,l+1}, v + \frac{1}{2}, w + \iota (t + \kappa \Delta t_{l+1}) \)
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Example: Cell $j, k$

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$$

Correction pass:

1. $\delta F_{j-\frac{1}{2},k}^{1,l+1} := -F_{j-\frac{1}{2},k}^{1,l}$

2. $\delta F_{j-\frac{1}{2},k}^{1,l+1} := \delta F_{j-\frac{1}{2},k}^{1,l+1} + \frac{1}{r_{l+1}^2} \sum_{\nu=0}^{r_{l+1}-1} F_{\nu+\frac{1}{2},w+\kappa}^{1,l+1} (t + \kappa \Delta t_{l+1})$

3. $\hat{Q}_{jk}^l(t + \Delta t_l) := Q_{jk}^l(t + \Delta t_l) + \frac{\Delta t_l}{\Delta x_l} \delta F_{j-\frac{1}{2},k}^{1,l+1}$
Refinement criteria

Scaled gradient of scalar quantity $w$

$$|w(Q_{j+1,k}) - w(Q_{jk})| > \epsilon_w, \ |w(Q_{j,k+1}) - w(Q_{jk})| > \epsilon_w, \ |w(Q_{j+1,k+1}) - w(Q_{jk})| > \epsilon_w$$
Common refinement criteria

Refinement criteria

Scaled gradient of scalar quantity \( w \)

\[ |w(Q_{j+1,k}) - w(Q_{jk})| > \epsilon_w, \quad |w(Q_{j,k+1}) - w(Q_{jk})| > \epsilon_w, \quad |w(Q_{j+1,k+1}) - w(Q_{jk})| > \epsilon_w \]

1. Richardson-type error estimation on interior cells
2. Create temporary Grid coarsened by factor 2
   Initialize with fine-grid-values of preceding time step
3. Compare temporary solutions

\[ \mathcal{H}^{\Delta t_l} Q'(t_l - \Delta t_l) = \mathcal{H}^{\Delta t_l} (\mathcal{H}^{\Delta t_l} Q'(t_l - \Delta t_l)) = \mathcal{H}^{\Delta t_l} Q'(t_l - \Delta t_l) \]

\[ \mathcal{H}^{2\Delta t_l} \bar{Q}'(t_l - \Delta t_l) \]
UML design of AMROC

- Classical framework approach with generic main program in C++
### UML design of AMROC

- Classical framework approach with generic main program in C++
- Customization / modification in Problem.h include file by derivation from base classes and redefining virtual interface functions
UML design of AMROC

- Classical framework approach with generic main program in C++
- Customization / modification in Problem.h include file by derivation from base classes and redefining virtual interface functions
- Predefined, scheme-specific classes provided for standard simulations
Multiresolution (MR) principles

- Multiresolution analysis is a tool to construct wavelet functions and consequently wavelet transforms
  - Information can be organized in different scale levels
  - Scale can be associated to periods bands
- Information in a certain level can be obtained by the combination of the coarser levels with the wavelet coefficient contributions and vice-versa

\[ Q^{\ell+1}_{\text{projection}} \iff Q_{\text{MR}}^{\ell+1} = \{ Q^\ell \} \cup \{ d^\ell \}, \]
Multiresolution (MR) principles

- Multiresolution analysis is a tool to construct wavelet functions and consequently wavelet transforms
  - Information can be organized in different scale levels
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\[
Q_{\ell + 1}^{\text{projection}} \iff \text{prediction} \quad Q_{\text{MR}}^{\ell + 1} = \{Q^\ell\} \cup \{d^\ell\},
\]

- PDE approach: Harten’s cell average MR is used, which is compatible with the underlying FV discretization [Rousell et al., 2003]
- Wavelet coefficients are used to characterize the local regularity of the solution
  - low amplitudes of the coefficients are associated to regions where the solution is smooth
  - high amplitudes appear only in regions where the solution is less regular.
MR operations for FV methods

1 Projection (restriction):

\[ P^\ell_{\ell+1} : Q^{\ell+1} \rightarrow Q^\ell \]

2 Prediction (prolongation):

\[ P^\ell_{\ell+1} : Q^\ell \rightarrow \tilde{Q}^{\ell+1} \]

\[
P^\ell_{\ell+1} : Q_i^\ell = \frac{1}{2} \left( Q_{2i}^{\ell+1} + Q_{2i+1}^{\ell+1} \right)
\]

\[
P^\ell_{\ell+1} : \tilde{Q}_{2i}^{\ell+1} = Q_i^\ell - \frac{1}{8} (Q_i^{\ell+1} - Q_i^{\ell-1})
\]

\[
P^\ell_{\ell+1} : \tilde{Q}_{2i+1}^{\ell+1} = Q_i^\ell + \frac{1}{8} (Q_{i+1}^{\ell} - Q_{i-1}^{\ell})
\]

2nd order polynomial interpolation as proposed by [Harten, 1995].
Use of wavelet transform for adaptation

Wavelet coefficients:

$$d^\ell = Q^{\ell+1} - P^{\ell+1} Q^\ell$$

Use of prediction error as refinement criterion:

$$|Q^\ell - P^\ell_{\ell-1} P^{\ell-1} Q^\ell| > \epsilon$$

Choice of $\epsilon$:
Use of wavelet transform for adaptation

Wavelet coefficients:

\[ d^\ell = Q^{\ell+1} - P^{\ell+1}_\ell Q^\ell \]

Use of prediction error as refinement criterion:

\[ |Q^\ell - P^\ell_{\ell-1} P^{\ell-1}_\ell Q^\ell| > \epsilon \]

Choice of \( \epsilon \):

- **level-independent** threshold parameter \( \epsilon \equiv \epsilon_\ell \)
- **Harten’s** thresholding strategy:

\[ \epsilon^\ell = \frac{\epsilon}{|\Omega|} 2^{2(\ell+1-L)}, \quad 0 \leq \ell < L \]

- **vector-valued** threshold in Euclidean norm of velocity field component of \( Q \)
Moving Gaussian bump

- **Initial condition:**

  \[
  \rho(x, y) = 1 + \exp \left( -\frac{x^2 + y^2}{16} \right), \quad u_x(x, y) = u_y(x, y) \equiv 1, \quad p(x, y) \equiv 1
  \]

- **Domain size:** \([-1, 1] \times [-1, 1]\)

- **Periodic boundary conditions**

- **The exact solution is a bump moving along the diagonal \(x = y\), without changing its shape.**

- **Base grid of** \(80 \times 80 + 3\) **levels (all refined by a factor 2)**

- **Finite volume scheme is the Van Leer flux-vector splitting, second order accurate MUSCL slope-limiting method combined with dimensional splitting.**

- **Clustering efficiency** \(\eta = 0.95\).

- **Final time:** \(t_e = 2\)
Moving Gaussian bump - refinement meshes

Hierarchical threshold

Gradient based

MR based, scalar

MR based, vector

Richardson estimation

MR based, scalar

MR based, vector

$\epsilon = \text{const.}$

MR based, scalar

MR based, vector
Cells on finest level versus error

\[ L^\rho_1 \]

\[ L^\rho_{1,AMR} \]

- Level-wise adaptation error: \( L_{1,AMR}(Q, G_\ell) = \sum_{i,j} |Q_{i,j} - Q^r_{i,j}| \Delta x_\ell \Delta y_\ell \).
- \( Q^r_{i,j} \) is reference solution from uniform at highest resolution.
Cells on finest level versus error

$L^\rho_1$

$L^\rho_{1,AMR}$

- **Level-wise adaptation error**: $L_{1,AMR}(Q, G_\ell) = \sum_{i,j} |Q_{i,j} - Q'_{i,j}| \Delta x_\ell \Delta y_\ell$.
  
  $Q'_{i,j}$ is reference solution from uniform at highest resolution

- Since the errors satisfy $L_1(Q) - L_{1,uni}(Q) \leq L_{1,AMR}(Q)$ and $L_{1,uni}$ is a constant, monotone behavior in $L_{1,AMR}$ will be preserved in $L_1$. 

R. Deiterding, M. O. Domingues – Design and application of wavelet-based refinement criteria
**Level-wise adaptation error:** \( L_{1,AMR}(Q, G_\ell) = \sum_{i,j} |Q_{i,j} - Q_{i,j}^r| \Delta x_\ell \Delta y_\ell \). 
\( Q_{i,j}^r \) is reference solution from uniform at highest resolution.

- Since the errors satisfy \( L_1(Q) - L_{1,uni}(Q) \leq L_{1,AMR}(Q) \) and \( L_{1,uni} \) is a constant, monotone behavior in \( L_{1,AMR} \) will be preserved in \( L_1 \).
- All MR criteria are more efficient than the SG and the Richardson estimation criteria.

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Lax–Liu configurations

Initial Values for the Lax-Liu configuration #6.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Domain position</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>Density ($\rho$)</td>
<td>1.00</td>
</tr>
<tr>
<td>Pressure ($p$)</td>
<td>1.00</td>
</tr>
<tr>
<td>Velocity component ($v_1$)</td>
<td>0.75</td>
</tr>
<tr>
<td>Velocity component ($v_2$)</td>
<td>-0.50</td>
</tr>
</tbody>
</table>

- 2nd-order accurate shock-capturing MUSCL-Hancock scheme with Minmod limiter and AUSMDV flux-vector splitting.
- Base mesh of $8 \times 8$ cells, with $8$ additional levels refined by factor $2$.
- Full mesh of $2048 \times 2048$ cells, final time of $t_e = 0.8$.
- Left: cluster threshold $\eta$ also varied. Total number of cells accumulated over all time steps.
Configuration #6 at $t_e = 0.8$ – Refinement

SAMR with SG criterion, $\epsilon^p = 0.05$

SAMR with MR criterion, $\epsilon = 0.0025$

1024$^2$ 2048$^2$ 4096$^2$
**Configuration #3 at** \( t_e = 0.3 \)

### Lax–Liu test cases

<table>
<thead>
<tr>
<th>Method</th>
<th>threshold</th>
<th># of cells (10^7)</th>
<th>( L_{1,AMR}^\rho ) (10^{-3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>SG</td>
<td>0.250</td>
<td>1.03</td>
<td>3.0</td>
</tr>
<tr>
<td>MR</td>
<td>0.005</td>
<td>6.42</td>
<td>1.9</td>
</tr>
<tr>
<td>MR*</td>
<td>0.005</td>
<td>6.83</td>
<td>1.6</td>
</tr>
<tr>
<td>MR+</td>
<td>0.010</td>
<td>7.59</td>
<td>2.3</td>
</tr>
<tr>
<td>MR*+</td>
<td>0.010</td>
<td>10.35</td>
<td>1.7</td>
</tr>
<tr>
<td>MRV</td>
<td>0.010</td>
<td>6.45</td>
<td>1.7</td>
</tr>
<tr>
<td>MRV*</td>
<td>0.010</td>
<td>6.76</td>
<td>1.5</td>
</tr>
<tr>
<td>MRV+</td>
<td>0.025</td>
<td>7.33</td>
<td>2.8</td>
</tr>
<tr>
<td>MRV*,+</td>
<td>0.025</td>
<td>7.55</td>
<td>2.3</td>
</tr>
</tbody>
</table>

V: Vector-valued threshold, *: hierarchical thresholding, +: one buffer cell

R. Deiterding, M. O. Domingues – Design and application of wavelet-based refinement criteria
We studied 19 configurations at 12 threshold values. For $\eta = 0.8$, the average cell savings of the MR approach versus SG are:

The majority of configurations involve all three wave types and for those the new MR criteria are most efficient.
Summary of Lax–Liu configuration tests

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- For the few configurations, that are dominated at large by isolated global discontinuities, especially #3, SG can be slightly more effective than MR.
Summary of Lax–Liu configuration tests

- We studied 19 configurations at 12 threshold values. For \( \eta = 0.8 \), the average cell savings of the MR approach versus SG are:

- The majority of configurations involve all three wave types and for those the new MR criteria are most efficient.
- For the few configurations, that are dominated at large by isolated global discontinuities, especially #3, SG can be slightly more effective than MR.
- The simple SG criterion is basically unaffected by numerical artefacts from the FV method, the MR criteria tend to over-refine those.
Ideal magneto-hydrodynamics simulation

**Governing equations**

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \left[ \rho \mathbf{u} \mathbf{u} + \left( p + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) \mathbf{I} - \mathbf{B}^t \mathbf{B} \right] &= 0 \\
\frac{\partial \rho E}{\partial t} + \nabla \cdot \left[ \left( \rho E + p + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) \mathbf{u} - (\mathbf{u} \cdot \mathbf{B}) \mathbf{B} \right] &= 0 \\
\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}^t \mathbf{B} - \mathbf{B}^t \mathbf{u}) &= 0
\end{align*}
\]

with equation of state

\[
p = (\gamma - 1) \left( \rho E - \rho \frac{u^2}{2} - \frac{B^2}{2} \right)
\]

The ideal MDH model is still hyperbolic, yet by re-writing the induction equation, one finds that the magnetic field has to satisfy at all times \( t \) the elliptic constraint

\[
\nabla \cdot \mathbf{B} = 0.
\]
Generalized Lagrangian multipliers for divergence control

Hyperbolic-parabolic correction of 2d ideal MHD model [Dedner et al., 2002]:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} = 0
\]

\[
\frac{\partial (\rho u_x)}{\partial t} + \frac{\partial}{\partial x} \left[ \rho u_x^2 + p \left( p + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) - B_x^2 \right] + \frac{\partial}{\partial y} \left( \rho u_x u_y - B_x B_y \right) = 0
\]

\[
\frac{\partial (\rho u_y)}{\partial t} + \frac{\partial}{\partial x} \left( \rho u_x u_y - B_x B_y \right) + \frac{\partial}{\partial y} \left[ \rho u_y^2 + p \left( p + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) - B_y^2 \right] = 0
\]

\[
\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x} \left[ \left( \rho E + p + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) u_x - (\mathbf{u} \cdot \mathbf{B}) B_x \right] + \frac{\partial}{\partial y} \left[ \left( \rho E + p + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) u_y - (\mathbf{u} \cdot \mathbf{B}) B_y \right] = 0
\]

\[
\frac{\partial B_x}{\partial t} + \frac{\partial}{\partial x} (u_y B_x - B_y u_x) + \frac{\partial}{\partial y} (u_y B_x - B_y u_x) = 0
\]

\[
\frac{\partial B_y}{\partial t} + \frac{\partial}{\partial x} (u_x B_y - B_x u_y) + \frac{\partial}{\partial y} (u_x B_y - B_x u_y) = 0
\]

\[
\frac{\partial B_z}{\partial t} + \frac{\partial}{\partial x} (u_x B_z - B_x u_x) + \frac{\partial}{\partial y} (u_y B_z - B_y u_z) = 0
\]

\[
\frac{\partial \psi}{\partial t} + c_h^2 \left( \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right) = \frac{4\pi}{c_p^2} \psi
\]
Orszag-Tang vortex

- Adaptive solution on 50 × 50 grid with 4 additional levels refined by $r_l = 2$
- Initial condition

\[
\begin{align*}
\rho(x, y, 0) &= \gamma^2, \quad u_x(x, y, 0) = -\sin(y), \quad u_y(x, y, 0) = \sin(x), \quad u_z(x, y, 0) = 0 \\
p(x, y, 0) &= \gamma, \quad B_x(x, y, 0) = -\sin(y), \quad B_y(x, y, 0) = 2\sin(x), \quad B_z(x, y, 0) = 0
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u_x(x, y, 0) &= -\sin(y), \\
u_y(x, y, 0) &= \sin(x), \\
u_z(x, y, 0) &= 0 \\
p(x, y, 0) &= \gamma, \\
B_x(x, y, 0) &= -\sin(y), \\
B_y(x, y, 0) &= 2\sin(x), \\
B_z(x, y, 0) &= 0
\end{align*}
\]
Orszag-Tang vortex

- Adaptive solution on $50 \times 50$ grid with 4 additional levels refined by $r_l = 2$
- Initial condition

\[
\begin{align*}
\rho(x, y, 0) &= \gamma^2, \quad u_x(x, y, 0) = -\sin(y), \quad u_y(x, y, 0) = \sin(x), \quad u_z(x, y, 0) = 0 \\
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Scaled gradient of $\rho$

Multi-resolution criterion with hierarchical thresholding
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Scaled gradient of $\rho$

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▶ Adaptive solution on $50 \times 50$ grid with 4 additional levels refined by $r_l = 2$

▶ Initial condition

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\rho(x, y, 0) &= \gamma^2, \\
\mathbf{u}(x, y, 0) &= -\sin(y), \\
\rho(x, y, 0) &= \gamma, \\
\mathbf{B}(x, y, 0) &= -\sin(y),
\end{align*}
\]

\[
\begin{align*}
u_x(x, y, 0) &= \sin(x), \\
u_y(x, y, 0) &= \sin(x), \\
u_z(x, y, 0) &= 0 \\
v_y(x, y, 0) &= 2\sin(x), \\
v_z(x, y, 0) &= 0
\end{align*}
\]
Orszag-Tang vortex

- Adaptive solution on 50 × 50 grid with 4 additional levels refined by $r_l = 2$
- Initial condition

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\rho(x, y, 0) = \gamma^2, \quad u_x(x, y, 0) = -\sin(y), \quad u_y(x, y, 0) = \sin(x), \quad u_z(x, y, 0) = 0
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\end{align*}
\]
Orszag-Tang vortex - cells on finest level vs. error

This is work in progress, and for now, the error is evaluated in $\rho$ only.

Compared are SG and MR with hierarchical threshold also applied to $\rho$ only.
Orszag-Tang vortex in 3D

- Adaptive solution on $32 \times 32 \times 32$ grid with 3 additional levels refined by $r_i = 2$
- Initial condition

$$\rho(x, y, z) = \gamma^2, \quad p(x, y, 0) = \gamma, \quad e = 0.2, \quad \gamma = 5/3, \quad u_z(x, y, z) = e \sin(2\pi z)$$

$$u_x(x, y, z) = -(1 + e \sin(2\pi z)) \sin(2\pi y), \quad u_y(x, y, z) = (1 + e \sin(2\pi z)) \sin(2\pi x)$$

$$B_x(x, y, z) = -\sin(2\pi y), \quad B_y(x, y, z) = \sin(4\pi x), \quad B_z(x, y, z) = 0$$
Ideal magneto-hydrodynamics simulation

Orszag-Tang vortex in 3D

- Adaptive solution on $32 \times 32 \times 32$ grid with 3 additional levels refined by $r_l = 2$
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Scaled gradients

Multi-resolution criteria with hierarchical thresholding
Orszag-Tang vortex in 3D

- Adaptive solution on $32 \times 32 \times 32$ grid with 3 additional levels refined by $r_l = 2$
- Initial condition

\[
\begin{align*}
\rho(x, y, z) &= \gamma^2, \quad p(x, y, 0) = \gamma, \quad e = 0.2, \quad \gamma = \frac{5}{3}, \quad u_z(x, y, z) = e \sin(2\pi z) \\
 u_x(x, y, z) &= -(1 + e \sin(2\pi z)) \sin(2\pi y), \quad u_y(x, y, z) = (1 + e \sin(2\pi z)) \sin(2\pi x) \\
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\end{align*}
\]
Orszag-Tang vortex in 3D

\[ t = 0.0 \]

- \( \rho \)
- \( B_x \)
- Divergence

Ideal magneto-hydrodynamics simulation
Orszag-Tang vortex in 3D

\[ t = 0.1 \]

\[ \rho \]  \hspace{2cm}  \[ B_x \]  \hspace{2cm}  \text{Divergence}
Orszag-Tang vortex in 3D

\[ t = 0.2 \]
Orszag-Tang vortex in 3D

\[ t = 0.3 \]
Orszag-Tang vortex in 3D

\[ t = 0.4 \]
Orszag-Tang vortex in 3D

$t = 0.5$
Ideal magneto-hydrodynamics simulation

Orszag-Tang vortex in 3D

\[ t = 0.5 \]

- Error is evaluated in \( \rho \) only
- SG and MR with hierarchical threshold applied to \( \rho, \rho u, \rho v \)
Conclusions

- For the first time, wavelet-based multi-resolution has been implemented as refinement criterion in a general and parallel structured AMR framework.
- An approach has been devised to quantify the efficiency of mesh adaptation criteria using the adaptation error for arbitrary problems.
Conclusions

▶ For the first time, wavelet-based multi-resolution has been implemented as refinement criterion in a general and parallel structured AMR framework.

▶ An approach has been devised to quantify the efficiency of mesh adaptation criteria using the adaptation error for arbitrary problems.

▶ Initial tests for shock-capturing FV method for Euler equations and ideal MHD equations are very promising:
  ▶ In complex configurations, involving discontinuities as well as rarefactions, the MR criterion is shown to be significantly more effective than currently used criteria.
  ▶ In rare situations, consisting primarily of global discontinuities, the SG criterion can be most efficient; however, the MR criterion can be tuned to give almost comparable performance.
Conclusions

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  ▶ In rare situations, consisting primarily of global discontinuities, the SG criterion can be most efficient; however, the MR criterion can be tuned to give almost comparable performance.

▶ Next steps will be to
  ▶ Replace the SAMR interpolation with the wavelet prediction for consistency (where possible)
  ▶ Test more complex MHD cases in combination with the MR criteria
References I


Clustering by signatures

\[\Delta = \gamma_{\nu+1} - 2\gamma_\nu + \gamma_{\nu-1}\]

Flagged cells per row/column

Technique from image detection: [Bell et al., 1994], see also [Berger and Rigoutsos, 1991], [Berger, 1986]
Clustering by signatures

\[ \gamma \]  Flagged cells per row/column
\[ \Delta \]  Second derivative of \( \gamma \), \( \Delta = \gamma_{\nu+1} - 2\gamma_{\nu} + \gamma_{\nu-1} \)

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Clustering by signatures

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\end{array}
\]

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\[
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\times & \times & & & & \\
\times & \times & & & & \\
\end{array}
\]

Second derivative of \( \Upsilon \), \( \Delta = \Upsilon_{\nu+1} - 2\Upsilon_{\nu} + \Upsilon_{\nu-1} \)

Technique from image detection: [Bell et al., 1994], see also [Berger and Rigoutsos, 1991], [Berger, 1986]
Clustering by signatures

Flagged cells per row/column

Second derivative of $\gamma$, $\Delta = \gamma_{\nu+1} - 2\gamma_{\nu} + \gamma_{\nu-1}$

Technique from image detection: [Bell et al., 1994], see also [Berger and Rigoutsos, 1991], [Berger, 1986]
Recursive generation of $\mathcal{G}_{l,m}$

1. 0 in $\Upsilon$
2. Largest difference in $\Delta$
3. Stop if ratio between flagged and unflagged cell $> \eta_{tol}$
Recursive generation of $\tilde{G}_{l,m}$

1. 0 in $\Gamma$
2. Largest difference in $\Delta$
3. Stop if ratio between flagged and unflagged cell $> \eta_{tol}$
Recursive generation of $\tilde{G}_{l,m}$

1. 0 in $\Upsilon$
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Parallelization

Rigorous domain decomposition

- Data of all levels resides on same node
- Grid hierarchy defines unique "floor-plan"
- Workload estimation

$$\mathcal{W}(\Omega) = \sum_{l=0}^{l_{\text{max}}} \left[ \mathcal{N}_l(G_l \cap \Omega) \prod_{\kappa=0}^{l} r_{\kappa} \right]$$

Parallel operations

- Synchronization of ghost cells
- Redistribution of data blocks within regridding operation
- Flux correction of coarse grid cells

Dynamic partitioning with space-filling curve

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